Problem. *Consider the collection of all subsets A of the topological space X. The operations of closure $A \mapsto \bar{A}$ and complementation $A \mapsto X - A$ are functions from this collection to itself.*

(a) Show that starting with a given set A, one can form no more than 14 distinct sets by applying these two operations successively.

(b) Find a subset A of $\mathbb{R}$ (in its usual topology) for which the maximum 14 is obtained.

This is a fun, relatively well-known problem posed, for instance, in Munkres's *Topology* (problem 21, §17, p. 102). There, it is credited to Kazimierz Kuratowski (1896-1980). I was really taken by the problem when I first read it. Its generality seemed quite remarkable. And where did this magical number 14 come from? I really recommend trying to solve it for yourself. My solution is on the following pages; it's very beautiful, but took me a few days to refine! Trust me, it's very fun. Even if you try and don't have the time to solve it, you will at least better appreciate how ultimately pretty it is.—*Carl McTague*

Hint 1. *Answer (b) first. Make the set as simple as possible and try to generalize the phenomena it exhibits. [Caution: the next hint is an answer to (b)]. Spend as much time on this step as you need, there's considerable cleverness and depth here.*

Hint 2. *What about the set* $A := (0,1) \cup (1,2) \cup [(3,4) \cap \mathbb{Q}] \cup \{5\}$*? Neat, isn't it? It feels like bigger, stranger spaces could exhibit more diverse phenomena, but this behavior is in fact general! Try to generalize what happens in* $\mathbb{R}$*. [Caution: the next hint is the last.]*

Hint 3 (Final)**.** *After solving (b), study the boundary points of A,* $\mathrm{Bd}(A) := \bar{A} \cap \overline{X - A}$. *Note that* $X = \mathrm{Int}(A) \sqcup \mathrm{Ext}(A) \sqcup \mathrm{Bd}(A)$ *(disjoint union). Do the boundary points fall into distinct classes? Can you study what happens to these classes under closure and complementation?*

Theorem. *Given a set A in an arbitrary topological space X, one can form no more than 14 distinct sets by closing and complementing the set successively.*

To prove the claim, I first classify the boundary points x of a set A, $\mathrm{Bd}(A) := \bar{A} \cap \overline{X - A}$, and establish a few essential results.

(1) $\forall U_x, U_x \cap \mathrm{Int}(A) \neq \emptyset$ and $U_x \cap \mathrm{Ext}(A) \neq \emptyset$. (Call such points *crisp*.)

(2) $\exists U_x$ s.t. $U_x \cap \mathrm{Int}(A) = \emptyset$ or $U_x \cap \mathrm{Ext}(A) = \emptyset$.

- (a) $\exists U_x$ s.t. $U_x \cap \mathrm{Int}(A) = U_x \cap \mathrm{Ext}(A) = \emptyset$. I.e. U_x contains only boundary points. (*fuzzy*)
- (b) $\exists U_x$ s.t. $U_x \cap \mathrm{Int}(A) = \emptyset$ and $U_x \cap \mathrm{Ext}(A) \neq \emptyset$. (*punctured exterior*)
- (c) $\exists U_x$ s.t. $U_x \cap \mathrm{Int}(A) \neq \emptyset$ and $U_x \cap \mathrm{Ext}(A) = \emptyset$. (*punctured interior*)

where U_x is short for a "neighborhood U_x of x."

From the very construction of this classification, it follows that:

Lemma 1. *If x is a boundary point, then it is precisely one of the following: crisp, fuzzy, punctured exterior or punctured interior.*

Furthermore, the following statements about classes of boundary points are true.

Lemma 2.

(1) *If x is crisp with respect to A, then it must be crisp with respect to both* $\mathrm{compl}(A)$ *and* $\mathrm{cl}(A)$.

(2) *If x is fuzzy with respect to A, then it must be fuzzy with respect to* $\mathrm{compl}(A)$ *and must be an interior point of* $\mathrm{cl}(A)$.

(3) *If x is punctured interior with respect to A, then it must be punctured exterior with respect to* $\mathrm{compl}(A)$ *and must be an interior point of* $\mathrm{cl}(A)$.

(4) *If x is punctured exterior with respect to A then it must be punctured interior with respect to* $\mathrm{compl}(A)$ *and must be* either *punctured exterior or crisp with respect to* $\mathrm{cl}(A)$.

Proof. Straightforward; the stage has been so well set! Perhaps (4) deserves special mention. Note that it is possible for a punctured exterior boundary point x to become crisp under closure; this may happen if so many nearby boundary points become interior points upon closure that every neighborhood about x contains at least one interior point. But it is also possible—in fact, seemingly more likely—that upon closure, there is still a neighborhood about x which contains no interior points. □

Figure 1 summarizes these results in graphical form. The arrow $M \xrightarrow{f} N$ is drawn if it is possible that x is a point of type N with respect to $f(A)$ given that x is a point of type M with respect to A. One can consider this graph as a nondeterministic finite automaton over the alphabet $\{\mathrm{compl}, \mathrm{cl}\}$—nondeterministic because the state "p. exterior" emits two transitions with label cl. The "subset construction" from automata theory may be used to convert this graph to a deterministic one—a graph in which each state emits at most a single arc for each symbol of the alphabet (this is accomplished, essentially, by inducing a graph on collections of original states). The result is shown in Figure 2. It may be interpreted

Lemma 3. *For any set A, the sets*

$$(\mathrm{cl} \circ \mathrm{compl} \circ \mathrm{cl})(A) \ \textit{and}$$

$$(\mathrm{cl} \circ \mathrm{compl} \circ \mathrm{cl} \circ \mathrm{compl})(A)$$

are closed sets, whose boundary points are all crisp.

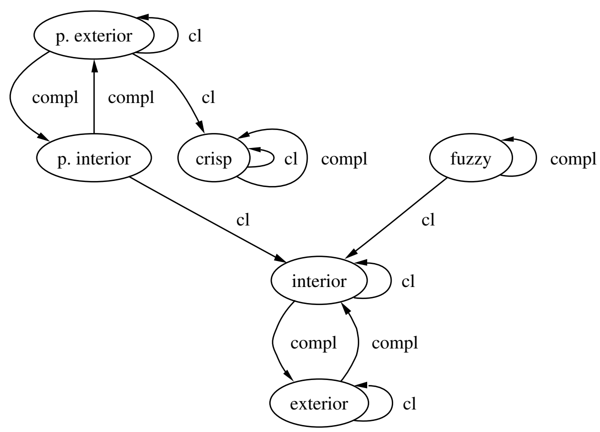

FIGURE 1. A graph of boundary point class results. The arrow $M \xrightarrow{f} N$ is drawn if it is possible that x is a point of type N with respect to $f(A)$ given that x is a point of type M with respect to A.

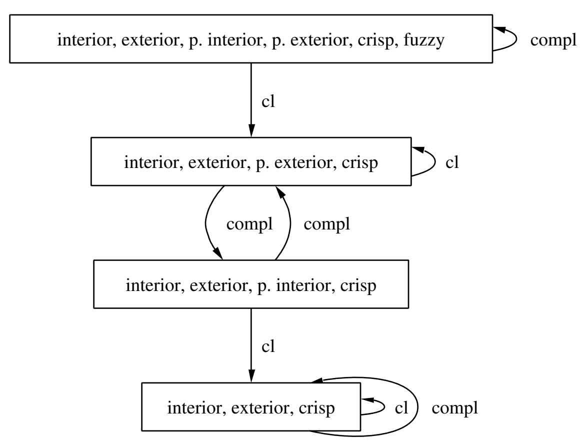

FIGURE 2. A "deterministic" version of the graph in Figure 1 obtained via the classic "subset construction" from automata theory. The arrow $M \xrightarrow{f} N$ is drawn if (x a point of one of the types in M with respect to A) $\Rightarrow$ (x a point of one of the types in N with respect to $f(A)$).

Proof. This follows immediately from the results summarized in Figures 1 and 2. Note that these are the shortest alternating words "accepted by" the automaton in Figure 2, assuming that the top state is the sole starting state and the bottom one, the sole accepting one. □

Note that since $\mathrm{compl}^2 = \mathrm{id}$ and $\mathrm{cl}^2 = \mathrm{cl}$, we need consider only alternating applications of compl and cl.

Lemma 4. *If all boundary points of a closed set C are crisp, then it generates at most 4 sets.*

Proof. Since C is closed,

$$C = \mathrm{Int}(C) \sqcup \mathrm{Bd}(C)$$

and it follows that

$$\begin{aligned}
\mathrm{compl}(C) =& \mathrm{Ext}(C) \\
(\mathrm{cl} \circ \mathrm{compl})(C) =& \mathrm{Ext}(C) \cup \mathrm{Bd}(\mathrm{compl}(C)) \\
=& \mathrm{Ext}(C) \sqcup \mathrm{Bd}(C) \\
(\mathrm{compl} \circ \mathrm{cl} \circ \mathrm{compl})(C) =& \mathrm{Int}(C) \\
(\mathrm{cl} \circ \mathrm{compl} \circ \mathrm{cl} \circ \mathrm{compl})(C) =& \mathrm{Int}(C) \cup \mathrm{Bd}((\mathrm{compl} \circ \mathrm{cl} \circ \mathrm{compl})(C)) \\
=& \mathrm{Int}(C) \sqcup \mathrm{Bd}(C) = C
\end{aligned}$$

since $\mathrm{cl}(A) = \mathrm{Int}(A) \cup \mathrm{Bd}(A)$ (c.f. problem 19, §17), since crisp boundary points are preserved under set closure and complementation and since the topological space X is the disjoint union $X = \mathrm{Int}(C) \sqcup \mathrm{Bd}(C) \sqcup \mathrm{Ext}(C)$. □

The theorem is established by combining the proceeding lemmas as illustrated in Figure 3.

Claim. *The subset*

$$A := (0, 1) \cup (1, 2) \cup [(3, 4) \cap \mathbb{Q}] \cup \{5\}$$

of $\mathbb{R}$ (with its usual topology) generates the maximum of 14 sets. So, the upper bound 14 is tight.

Carl McTague

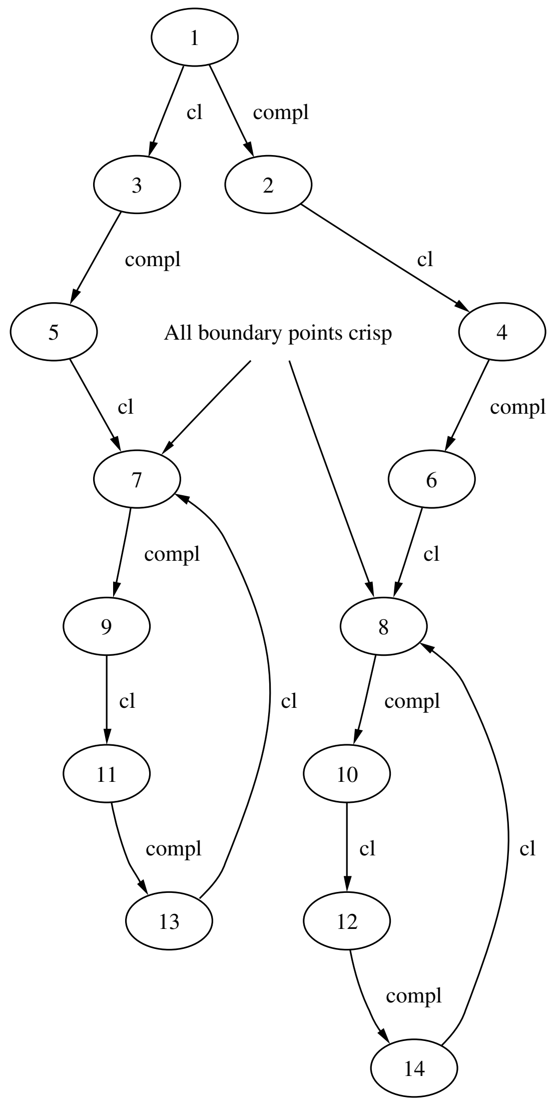

FIGURE 3. A graph of the possible sets generated from a single set "1." Note that we may count the original set "1" among the generated ones, since $\text{compl}^2 = \text{id}$.