A New Approach to
Euler Calculus for Continuous Integrands

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Gratitude
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Euler Calculus

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But this is only true for finite unions so \( \chi \) isn't a true measure.

As a result the Euler integral of a “simple function” is easy to define:

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The aim is to construct an integration theory *using the Euler characteristic* $\chi$ *as measure*.

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But it behaves poorly under limits:  
\[ \lim s_n = \lim s'_n \text{ doesn’t necessarily imply that} \]
\[ \lim \int s_n \, d\chi = \lim \int s'_n \, d\chi \]
Baryshnikov-Ghrist studied this failure of convergence.

They considered the Euler integrals of two sequences of simple functions approaching a given continuous function \( \alpha \):

\[
\int \alpha \left[ \, d\chi \right] = \lim_{n \to \infty} \frac{1}{n} \int [n\alpha] \, d\chi \quad \int \alpha \left[ \, d\chi \right] = \lim_{n \to \infty} \frac{1}{n} \int [n\alpha] \, d\chi
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*Ex:*

$$\int 1_{[0,1]} [d\chi] = \alpha(1)$$

$$\int 1_{[0,1]} [d\chi] = \alpha(0)$$

Although these integrals differ, they are in a sense dual.
A naive starting point

**Lemma** (Baryshnikov-Ghrist): If $\alpha : \Delta^i \to \mathbb{R}$ is affine then:

\[
\int_{\text{int}(\Delta)} \alpha[d\chi] = (-1)^i \inf \alpha \quad \int_{\text{int}(\Delta)} \alpha[d\chi] = (-1)^i \sup \alpha
\]

It follows immediately that neither integral is additive.
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**Lemma** (Baryshnikov-Ghrist): If $\alpha : \Delta^i \to \mathbb{R}$ is affine then:

$$\int_{\text{int}(\Delta)} \alpha \, d\chi = (-1)^i \inf \alpha \quad \int_{\text{int}(\Delta)} \alpha \, d\chi = (-1)^i \sup \alpha$$

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This made me wonder whether **maybe the answer should instead be**:

$$\int_{\text{int}(\Delta)} \alpha \, d\chi = (-1)^i \alpha(\hat{\Delta})$$

where $\hat{\Delta}$ is the barycenter of $\Delta$. At least then it would be **additive**.
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**Tentative Definition:** For $X$ and $\alpha : X \to \mathbb{R}$ simplicial, let:

$$\int_X \alpha \, d\chi = \sum_{\Delta^i \in X} (-1)^i \alpha(\hat{\Delta})$$
Exploration of the integral’s properties

It is *not* invariant under subdivision.

Ex:

\[
\int \alpha \, d\chi = \alpha(\Delta) = \frac{1}{3} \sum \alpha(v_i)
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\[ \int \alpha \, d\chi = \alpha(\hat{\Delta}) = \frac{1}{3} \sum \alpha(v_i) \]
\[ \int \alpha \, d\chi = \frac{1}{6} \alpha(v_0) + \left( \frac{1}{3} + \frac{1}{6} \lambda \right) \alpha(v_1) + \left( \frac{1}{2} - \frac{1}{6} \lambda \right) \alpha(v_2) \]

These integrals differ for any \( 0 \leq \lambda \leq 1. \)

*(We shall return to this example later.)*
Exploration of the integral’s properties, cont’d

But if one carries out a full barycentric subdivision then, after considerable calculation, one recovers the original integral.

\[ \int_{\Delta} \alpha^{(1)} \, d\chi = \alpha(\hat{\Delta}) = \int_{\Delta} \alpha \, d\chi \]
Exploration of the integral’s properties, cont’d

But if one carries out a full barycentric subdivision then, after considerable calculation, one recovers the original integral.

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**Theorem:** For any \( n \geq 1 \):

\[ \int_X \alpha \, d\chi = \int_{X^{(n)}} \alpha^{(n)} \, d\chi \]

where \( \alpha^{(n)} : X^{(n)} \to \mathbb{R}^{(n)} \) is the linear extension of \( \alpha \) to the \( n \)th barycentric subdivision \( X^{(n)} \) of \( X \).

*(This result appears in retrospect to have been a distraction though.)*
Rewriting the sum

The integral may be rewritten:

\[ \int_X \alpha \, d\chi = \sum_{\Delta^i \in X} (-1)^i \alpha(\hat{\Delta}) = \sum_v \alpha(v)w(v) \]

where \( v \) ranges over each vertex of \( X \) and where:

\[ w(v) = \sum_i (-1)^i \frac{1}{i+1} \# \{ i\text{-simplices containing } v \} \]
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We next interpret the number \(w(v)\) geometrically.
Banchoff’s 1967 work on curvature of embedded polyhedra

Let $X$ be a simplicial complex embedded in $\mathbb{R}^n$.

**Def** (Banchoff): The *curvature* at a vertex $v$ of $X$ is:

$$\kappa(v) = \sum_{\Delta^i \in X} (-1)^i \mathcal{E}(\Delta^i, v)$$

where the excess angle $\mathcal{E}(\Delta^i, v)$ at $v$ of a simplex $\Delta^i \subset \mathbb{R}^i$ is:

$$\mathcal{E}(\Delta^i, v) = \frac{1}{\text{vol}(S^{i-1})} \int_{S^{i-1}} \left[ \langle \xi, v \rangle \geq \langle \xi, x \rangle \text{ for all } x \text{ in } \Delta^i \right] d\xi$$

where $\xi$ ranges over the unit sphere $S^{i-1} \subset \mathbb{R}^i$, and $[P] = \begin{cases} 1 & \text{if } P \\ 0 & \text{if } \neg P \end{cases}$ is the Iverson bracket.
Geometric interpretation of $w(v)$

**Def:** Given a simplicial complex $X$, let $d_X$ be the intrinsic metric which makes each simplex flat and gives each 1-simplex length 1.
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**Theorem:** $w(v) = \kappa(v)$ if one gives $X$ the metric $d_X$. 

Ex: This explains why the integral isn't invariant under subdivision:

$k \neq 0$

$= 1$

Should have integrated like this but integrated like this instead.
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Improved definition of integral

So the integral we’re after depends on the metric structure of the domain—not just its topology.
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**Correct Definition:** For a metric simplicial complex $X$ and a simplicial map $\alpha : X \to \mathbb{R}$, let:

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**Correct Definition:** For a metric simplicial complex $X$ and a simplicial map $\alpha : X \to \mathbb{R}$, let:

$$\int_X \alpha \, d\chi = \sum_v \alpha(v) \kappa(v)$$

i.e. *Euler integration is integration with respect to curvature.*

This makes a lot of sense actually...
Chern’s 1945 work on the Gauss-Bonnet theorem

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$$\int_M \text{Pf}(\Omega) = \chi(M)$$

That is, curvature is *infinitesimal Euler characteristic*. 
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**Chern-Gauss-Bonnet Thm:** For a compact Riemannian manifold \( M \) with boundary \( \partial M \):

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**Simplicial Chern-Gauss-Bonnet Thm** (Banchoff):

$$\sum_v \kappa(v) = \chi(X)$$

(Note that Banchoff’s work applies to *singular* spaces.)
The importance of the boundary contribution

Chern-Gauss-Bonnet only applies to *compact* spaces, so *one should only integrate curvature over compact domains.*
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Chern-Gauss-Bonnet only applies to \textit{compact} spaces, so \textbf{one should only integrate curvature over compact domains.}

\textbf{Ex:} An open interval $X = (0, 1)$ has curvature 0 yet has $\chi(X) = -1$. But if we write:

$$\int 1_{(0,1)} \, d\chi = \int (1_{[0,1]} - 1_{\emptyset} - 1_{\{1\}}) \, d\chi$$

then we can use curvature integration to correctly compute:

$$= (1/2 + 1/2) - 1 - 1 = -1$$
Curvature is as general as Euler characteristic —i.e. it can be defined within any “O-minimal theory”.
Bröcker-Kuppe’s 2000 work on curvature of stratified spaces

Bröcker-Kuppe used Goresky-MacPherson’s work on stratified Morse theory to define *curvature for any “tame” stratified space.*

(This includes all spaces in an O-minimal theory.)
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**Stratified Morse theory**
Loosely speaking, a Morse function $f : X \to \mathbb{R}$ on a stratified space $X$ is one which restricts to a classical Morse function on each stratum.
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\[
(P, Q) = B(x, \delta) \cap \left( f^{-1}[f(x) - \epsilon, f(x) + \epsilon], f^{-1}[f(x) - \epsilon] \right)
\]

where \( B(x, \delta) \) is a closed ball of radius \( \delta \) centered at \( x \).
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**Remark:** \( P \) is always a cone so \( \chi(P, Q) = \chi(P) - \chi(Q) = 1 - \chi(Q) \).
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Definition (Bröcker-Kuppe): The curvature measure $\kappa_X(U)$ of a Borel set $U \subseteq X$ is:

$$\kappa_X(U) = \frac{1}{\text{vol}(S^{N-1})} \int_{S^{N-1}} \sum_{x \in U} \left( 1 - \chi(B(x, \delta) \cap f^{-1}[f(x) - \epsilon]) \right) d\xi$$

where $\xi$ ranges over the unit sphere $S^{N-1} \subseteq \mathbb{R}^N$. 

Remark (Bröcker-Kuppe): If $X$ is “tame” then $f(x) = \langle \xi, x \rangle$ is a stratified Morse function for $dS^{N-1}$ almost all.

Remark: If $X$ is a simplicial complex then the curvature measure is concentrated at the vertices, where it agrees with Banchoff’s ($v$).
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where $\xi$ ranges over the unit sphere $S^{N-1} \subset \mathbb{R}^N$.

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Example from Bröcker & Kuppe’s 2000 paper
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\[(2\pi)^{-1} d l\]
Euler Integration for Stratified Spaces

**Stratified Gauss-Bonnet Thm** (Bröcker-Kuppe):
If $X$ is compact then $\chi(X) = \kappa_X(X)$, that is:

$$\int 1_X \, d\chi = \int 1_X \, d\kappa_X$$
Euler Integration for Stratified Spaces

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So we reach our:

**Final Definition:** For a compact tame stratified space $X \subset \mathbb{R}^N$ and a continuous function $\alpha : X \to \mathbb{R}$, let:

$$\int_X \alpha \, d\chi = \int_X \alpha \, d\kappa_X$$

where the right hand side is Lebesgue integration with respect to the Bröcker-Kuppe curvature measure $\kappa_X$. 
The standard Fubini theorem therefore applies:

**Fubini Thm:** If \( f : Y \times Z \to Y \) is the projection then \( \kappa_{Y \times Z} \cong \kappa_Y \times \kappa_Z \) and:

\[
\int_{Y \times Z} \alpha \, d\kappa_{Y \times Z} = \int_Y \left( \int_Z \alpha \, d\kappa_Z \right) \, d\kappa_Y
\]
Functoriality

For simple functions, Euler integration extends to a functor:

\[ E : \text{spaces} \rightarrow \text{abelian groups} \]

\[ X \mapsto \text{E}(X) = \{ \text{simple functions on } X \} \]

\[ X \xrightarrow{f} Y \mapsto \text{group homomorphism } \text{E}(X) \xrightarrow{\text{E}(f)} \text{E}(Y) \text{ defined by:} \]

\[ \text{E}(f)(1_W)(y) = \chi(1_W \cap f^{-1}(y)) \]
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In other words, integration over the fiber is functorial.

(Aside: MacPherson’s theory of Chern classes for singular varieties is a natural transformation \( E \to H_*(-, \mathbb{Z}) \).)
Functoriality, cont’d

Functoriality is less straightforward for continuous integrands.
Functoriality, cont’d

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A first idea is to let:

\[ \tilde{E}(X) = \left\{ \sum_{\text{finite}} \alpha_i \mid \alpha_i : K_i \to \mathbb{R} \text{ continuous}, K_i \subset X \text{ compact} \right\} \]

Euler integration works well for these functions.
Functoriality, cont’d

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Euler integration works well for these functions.

But there are problems defining a pushforward \( \tilde{E}(f) : \tilde{E}(X) \to \tilde{E}(Y) \).

One could optimistically define:

\[ \tilde{E}(f)(\alpha) = \left[ \frac{d(f_*(\alpha \cdot \kappa_X))}{d\kappa_Y} \right] \]

Functoriality would then follow from the chain rule.

But this derivative generally doesn’t exist.
Example of $f : X \rightarrow Y$ where the derivative $\left[ \frac{df_*(\kappa_X)}{d\kappa_Y} \right]$ doesn’t exist:

\[ \text{Graph of } \left[ \frac{df_*(\kappa_X)}{dx} \right] = \frac{1}{\pi \sqrt{1-x^2}} \]
Example of $f : X \to Y$ where the derivative $\left[ \frac{df_* (\kappa_X)}{d\kappa_Y} \right]$ doesn’t exist:

Since $\kappa_Y$ is concentrated at the two ends, the Lebesgue decomposition must look like:

$$f_* (\kappa_X) = f_* (\kappa_X) |_{\kappa_Y} \kappa_Y + f_* (\kappa_X)_{\perp \kappa_Y} = 0$$

Graph of $\left[ \frac{df_* (\kappa_X)}{dx} \right] = \frac{1}{\pi \sqrt{1-x^2}}$
Functoriality via measures

So to define a functor, need to consider not **functions** but **measures**:

\[ X \ni \tilde{\mathcal{E}}(X) = \{ \text{signed measures on } X \} \]

\[ X \xrightarrow{f} Y \ni \tilde{\mathcal{E}}(f) : \tilde{\mathcal{E}}(X) \rightarrow \tilde{\mathcal{E}}(Y) \text{ defined by:} \]

\[ \tilde{\mathcal{E}}(f)(\mu) = f_*(\mu), \text{ the pushforward measure} \]
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**Remark:** For each space \( X \) there is a homomorphism \( E(X) \to \tilde{E}(X) \) sending \( 1_K \mapsto \kappa_K \) for \( K \subset X \) compact but, as the preceding example shows, these do not fit into a natural transformation although pushforward to a point always commutes:

\[
\begin{array}{ccc}
E(X) & \longrightarrow & \tilde{E}(X) \\
\downarrow & & \downarrow \\
E(\text{pt}) & = & \tilde{E}(\text{pt})
\end{array}
\]
A generalization of the Fubini theorem

By the earlier Fubini theorem, pushforward agrees with integration over the fiber for (metric) fiber bundles.
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"Theorem": Under "fairly general conditions":

$$f_*(\alpha \cdot \kappa_X) = \frac{1}{\chi(f^{-1}(y))} \int_{f^{-1}(y)} \alpha \, d\kappa_{f^{-1}(y)} \cdot f_*(\kappa_X)$$
Interpolating between Baryshnikov-Ghrist’s non-additive but dual:

\[ \int_X \alpha \, [d\chi] \quad \int_X \alpha \, [d\chi] \]

leads to an additive self-dual integral, and this integral is integration with respect to curvature:

\[ \int_X \alpha \, d\kappa_X \]

This integral is as general as the Euler characteristic itself.

In order to extend this integral to a functor, one must rely on the pushforward of measures.
References