A New Approach to

Euler Calculus for Continuous Integrands

Carl McTague

mctague.org/carl

University of Southampton

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from August Johns Hopkins University

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But it behaves poorly under limits:

$\lim s_n = \lim s'_n$ doesn't necessarily imply that

$$\lim \int s_n \, \mathrm{d}\chi = \lim \int s'_n \, \mathrm{d}\chi$$

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Baryshnikov-Ghrist studied this failure of convergence.

They considered the Euler integrals of two sequences of simple functions approaching a given continuous function α:

$$\int \alpha \lfloor d\chi \rfloor = \lim_n \frac{1}{n} \int \lfloor n\alpha \rfloor \, d\chi \qquad \int \alpha \lceil d\chi \rceil = \lim_n \frac{1}{n} \int \lceil n\alpha \rceil \, d\chi$$

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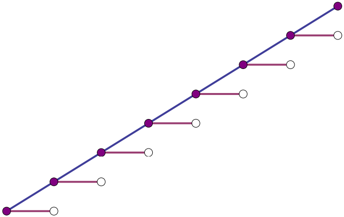
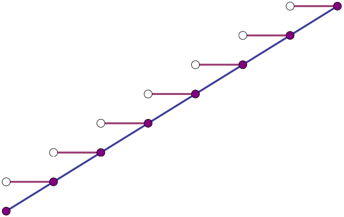
Ex:



$$\int 1_{[0,1]} \lfloor d\chi \rfloor = \alpha(1)$$



$$\int 1_{[0,1]} \lceil d\chi \rceil = \alpha(0)$$

Although these integrals differ, they are in a sense dual.

A naive starting point

Lemma (Baryshnikov-Ghrist): If $\alpha : \Delta^i \to \mathbf{R}$ is affine then:

$$\int_{\mathrm{int}(\Delta)} \alpha \lfloor d\chi \rfloor = (-1)^i \inf \alpha \qquad \int_{\mathrm{int}(\Delta)} \alpha \lceil d\chi \rceil = (-1)^i \sup \alpha$$

It follows immediately that neither integral is additive.

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This made me wonder whether **maybe the answer *should instead be:***

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Tentative Definition: For X and $\alpha : X \to \mathbf{R}$ *simplicial*, let:

$$\int_X \alpha \, \mathrm{d}\chi = \sum_{\Delta^i \in X} (-1)^i \alpha(\hat{\Delta})$$

Exploration of the integral's properties

It is *not* invariant under subdivision.

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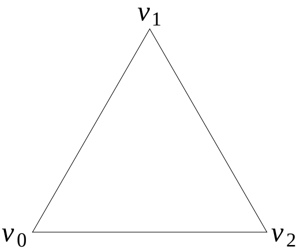

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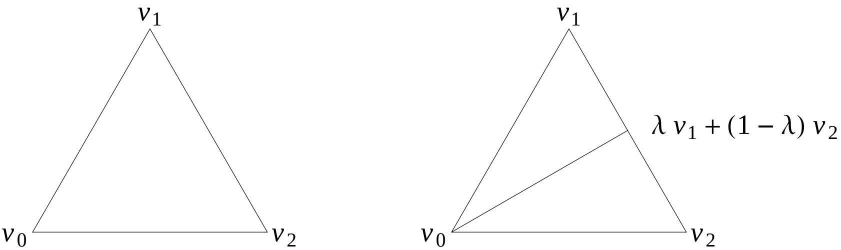


$\int \alpha \, \mathrm{d}\chi = \alpha(\hat{\Delta}) = \frac{1}{3} \sum \alpha(v_i)$

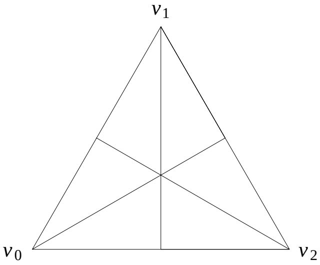
$$\int \alpha \, \mathrm{d}\chi = \frac{1}{6}\alpha(v_0) + \left(\frac{1}{3} + \frac{1}{6}\lambda\right) \alpha(v_1) + \left(\frac{1}{2} - \frac{1}{6}\lambda\right) \alpha(v_2)$$

These integrals differ for any $0 \leq \lambda \leq 1$.

(We shall return to this example later.)

Exploration of the integral's properties, cont'd

But if one carries out a full barycentric subdivision then, after considerable calculation, one recovers the original integral.



$$\int_{\Delta^{(1)}} \alpha^{(1)}\,\mathrm{d}\chi = \alpha(\hat{\Delta}) = \int_{\Delta} \alpha\,\mathrm{d}\chi$$

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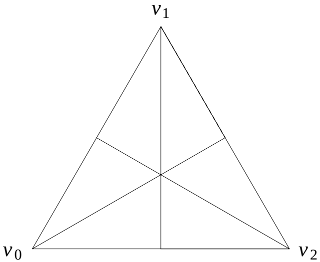

$$\int_{\Delta^{(1)}} \alpha^{(1)}\, \mathrm{d}\chi = \alpha(\hat{\Delta}) = \int_{\Delta} \alpha\, \mathrm{d}\chi$$

Theorem: For any $n \geq 1$:

$$\int_X \alpha\, \mathrm{d}\chi = \int_{X^{(n)}} \alpha^{(n)}\, \mathrm{d}\chi$$

where $\alpha^{(n)} : X^{(n)} \to \mathbf{R}^{(n)}$ is the linear extension of α to the nth barycentric subdivision $X^{(n)}$ of X.

(This result appears in retrospect to have been a distraction though.)

Rewriting the sum

The integral may be rewritten:

$$\int_X \alpha \, \mathrm{d}\chi = \sum_{\Delta^i \in X} (-1)^i \alpha(\hat{\Delta}) = \sum_v \alpha(v) \mathrm{w}(v)$$

where v ranges over each vertex of X and where:

$$\mathrm{w}(v) = \sum_i (-1)^i \frac{1}{i+1} \, \#\{i\text{-simplices containing } v\}$$

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We next interpret the number $\mathrm{w}(v)$ *geometrically.*

Banchoff's 1967 work on curvature of embedded polyhedra

Let X be a simplicial complex *embedded in* $\mathbf{R}^n$.

Def (Banchoff): The *curvature* at a vertex v of X is:

$$\kappa(v) = \sum_{\Delta^i \in X} (-1)^i \mathcal{E}(\Delta^i, v)$$

where the excess angle $\mathcal{E}(\Delta^i, v)$ at v of a simplex $\Delta^i \subset \mathbf{R}^i$ is:

$$\mathcal{E}(\Delta^i, v) = \frac{1}{\mathrm{vol}(S^{i-1})} \int_{S^{i-1}} \left[\langle \xi, v \rangle \geq \langle \xi, x \rangle \text{for all } x \text{ in } \Delta^i \right] \mathrm{d}\xi$$

where ξ ranges over the unit sphere $S^{i-1} \subset \mathbf{R}^i$, and $[P] = \begin{cases} 1 & \text{if } P \\ 0 & \text{if } \neg P \end{cases}$

is the Iverson bracket.

Geometric interpretation of $\mathrm{w}(v)$

Def: Given a simplicial complex X, let d_X be the intrinsic metric which makes each simplex flat and gives each 1-simplex length 1.

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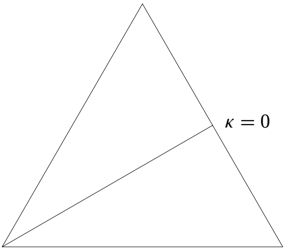
Ex: This explains why the integral isn't invariant under subdivision:



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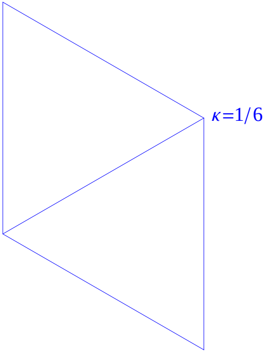

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Improved definition of integral

So the integral we're after *depends on the metric structure of the domain*—not just its topology.

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Correct Definition: For a metric simplicial complex X and a simplicial map $\alpha : X \to \mathbf{R}$, let:

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i.e. ***Euler integration is integration with respect to curvature***.

This makes a lot of sense actually...

Chern's 1945 work on the Gauss-Bonnet theorem

Chern-Gauss-Bonnet Thm: For a compact Riemannian manifold M:

$$\int_M \mathrm{Pf}(\Omega) = \chi(M)$$

That is, **curvature is *infinitesimal Euler characteristic***.

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Simplicial Chern-Gauss-Bonnet Thm (Banchoff):

$$\sum_v \kappa(v) = \chi(X)$$

(Note that Banchoff's work applies to ***singular*** *spaces.)*

The importance of the boundary contribution

Chern-Gauss-Bonnet only applies to *compact* spaces,

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Ex: An open interval $X = (0, 1)$ has curvature 0 yet has $\chi(X) = -1$.

But if we write:

$$\int 1_{(0,1)} \, d\chi = \int \left(1_{[0,1]} - 1_{\{0\}} - 1_{\{1\}}\right) d\chi$$

then we can use curvature integration to correctly compute:

$$= (1/2 + 1/2) - 1 - 1 = -1$$

Curvature is as general as Euler characteristic

—i.e. it can be defined within any "O-minimal theory".

Bröcker-Kuppe's 2000 work on curvature of stratified spaces

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$$(P, Q) = \mathrm{B}(x, \delta) \cap \left(f^{-1}[f(x) - \epsilon, f(x) + \epsilon],\ f^{-1}[f(x) - \epsilon] \right)$$

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Remark: P is always a cone so $\chi(P, Q) = \chi(P) - \chi(Q) = 1 - \chi(Q)$.

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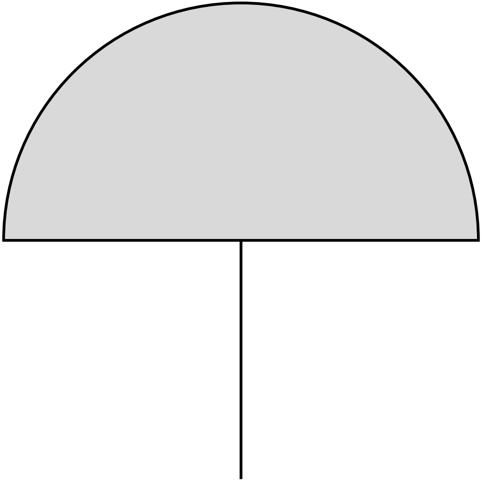
Remark: If X is a simplicial complex then the curvature measure is concentrated at the vertices, where it agrees with Banchoff's $\kappa(v)$.

Example from Bröcker & Kuppe's 2000 paper



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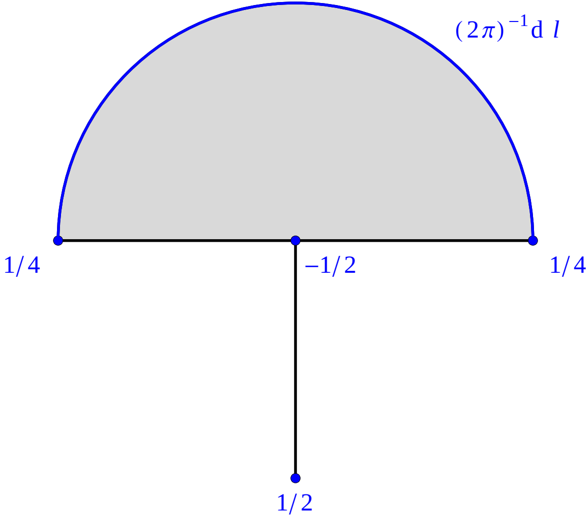

Euler Integration for Stratified Spaces

Stratified Gauss-Bonnet Thm (Bröcker-Kuppe):

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So we reach our:

Final Definition: For a compact tame stratified space $X \subset \mathbf{R}^N$ and a continuous function $\alpha : X \to \mathbf{R}$, let:

$$\int_X \alpha \, \mathrm{d}\chi = \int_X \alpha \, \mathrm{d}\kappa_X$$

where the right hand side is Lebesgue integration with respect to the Bröcker-Kuppe curvature measure κ_X.

Fubini theorem

The standard Fubini theorem therefore applies:

Fubini Thm: If $f : Y \times Z \to Y$ is the projection then $\kappa_{Y\times Z} \cong \kappa_Y \times \kappa_Z$ and:

$$\int_{Y\times Z} \alpha \, \mathrm{d}\kappa_{Y\times Z} = \int_Y \left(\int_Z \alpha \, \mathrm{d}\kappa_Z \right) \mathrm{d}\kappa_Y$$

Functoriality

For simple functions, Euler integration extends to a functor:

$$
\begin{array}{rcl}
\mathrm{E} : \text{spaces} & \to & \text{abelian groups} \\
X & \mapsto & \mathrm{E}(X) = \{\text{simple functions on } X\} \\
X \xrightarrow{f} Y & \mapsto & \text{group homomorphism } \mathrm{E}(X) \xrightarrow{\mathrm{E}(f)} \mathrm{E}(Y) \text{ defined by:} \\
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(**Aside:** MacPherson's theory of Chern classes for singular varieties is a natural transformation $\mathrm{E} \to \mathrm{H}_*(-, \mathbf{Z})$.)

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$$\tilde{\mathrm{E}}(X) = \left\{ \sum_{\text{finite}} \alpha_i \;\middle|\; \alpha_i : K_i \to \mathbf{R} \text{ continuous}, K_i \subset X \text{ compact} \right\}$$

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Euler integration works well for these functions.

But there are problems defining a pushforward $\tilde{\mathrm{E}}(f) : \tilde{\mathrm{E}}(X) \to \tilde{\mathrm{E}}(Y)$.

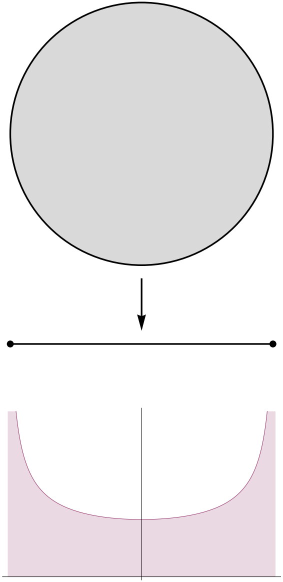
One could optimistically define:

$$\tilde{\mathrm{E}}(f)(\alpha) = \left[\frac{\mathrm{d}(f_*(\alpha \cdot \kappa_X))}{\mathrm{d}\kappa_Y} \right] \longleftarrow \text{the Radon-Nikodym derivative}$$

Functoriality would then follow from the chain rule.

But this derivative generally doesn't exist.

Example of $f : X \to Y$ where the derivative $\left[\frac{\mathrm{d}f_*(\kappa_X)}{\mathrm{d}\kappa_Y}\right]$ doesn't exist:



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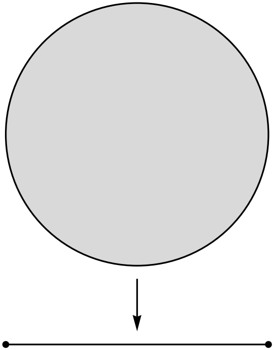
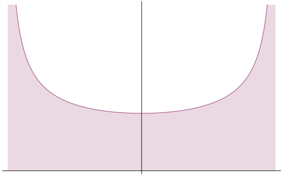


Since κ_Y is concentrated at the two ends, the Lebesgue decomposition must look like:

$$f_*(\kappa_X) = \underbrace{f_*(\kappa_X)^{\|\kappa_Y}}_{=0} \kappa_Y + f_*(\kappa_X)^{\perp\kappa_Y}$$



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So to define a functor, need to consider not **functions** but **measures**:

$$\begin{array}{rcl} X & \mapsto & \tilde{\mathrm{E}}(X) = \{\text{signed measures on } X\} \\ X \xrightarrow{f} Y & \mapsto & \tilde{\mathrm{E}}(f) : \tilde{\mathrm{E}}(X) \to \tilde{\mathrm{E}}(Y) \text{ defined by:} \\ & & \tilde{\mathrm{E}}(f)(\mu) = f_*(\mu), \text{ the pushforward measure} \end{array}$$

Remark: For each space X there is a homomorphism $\mathrm{E}(X) \to \tilde{\mathrm{E}}(X)$ sending $1_K \mapsto \kappa_K$ for $K \subset X$ compact but, as the preceding example shows, these do not fit into a natural transformation although pushforward to a point always commutes:

$$\begin{array}{ccc} \mathrm{E}(X) & \longrightarrow & \tilde{\mathrm{E}}(X) \\ \downarrow & & \downarrow \\ \mathrm{E}(\mathrm{pt}) & = & \tilde{\mathrm{E}}(\mathrm{pt}) \end{array}$$

A generalization of the Fubini theorem

By the earlier Fubini theorem, pushforward agrees with integration over the fiber for (metric) fiber bundles.

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"**Theorem**": Under "fairly general conditions":

$$f_*(\alpha \cdot \kappa_X) = \frac{1}{\chi\big(f^{-1}(y)\big)} \int_{f^{-1}(y)} \alpha \, \mathrm{d}\kappa_{f^{-1}(y)} \cdot f_*(\kappa_X)$$

Summary

Interpolating between Baryshnikov-Ghrist's non-additive but dual:

$$\int_X \alpha \lfloor \mathrm{d}\chi \rfloor \qquad\qquad \int_X \alpha \lceil \mathrm{d}\chi \rceil$$

leads to an additive self-dual integral, and this integral is integration with respect to curvature:

$$\int_X \alpha \, \mathrm{d}\kappa_X$$

This integral is as general as the Euler characteristic itself.

In order to extend this integral to a functor, one must rely on the pushforward of measures.

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