Topology — the mathematics of spaces

Classical Question: Can $\mathbb{C}^n$ be deformed into $\mathbb{C}$? Answer: No. But how are we sure?

Because the "Euler Characteristic $X"$ detects:

$$X(\mathbb{C}^n) = 2 \neq 0 = X(\mathbb{C})$$

In fact, $X$ classifies all (compact, oriented) 2-dimensional surfaces:

$$X\left(\begin{array}{c}
\includegraphics{surface}\n\end{array}\right) = 2 - 2g$$

Higher-dimensional surfaces are much more difficult to classify.
I am studying the "signature" $\sigma$.

It is a "cobordism invariant". (It is not.)

This means that if two spaces $M$ and $N$ "cobound" a space $W$:

\[ M \sqcup W \sqcup N \]

then $\sigma(M) = \sigma(N)$.

Your pair of pants, for example, say that

$\sigma(\text{circle}) = \sigma(2 \text{ circles}) = 2 \cdot \sigma(\text{circle})$

imply $\sigma(\text{circle}) = 0$

Cobordism provides a stepping stone in classifying spaces: first classify "up to cobordism", then refine the classification.
This is all old news. What am I studying?
The classification of spaces with singularities.

\[ \text{In particular, I am studying how to compute the signature of such spaces.} \]
\[ \text{(It is surprisingly difficult and involves the "derived category of sheaf complexes".)} \]

The hope is to then use this to classify singular spaces— at least "up to cobordism."
What is the Euler Characteristic $X$?

It's the "alternating sum" of the number of "cells" in a space:

\[
\begin{array}{ccc}
0\text{-cells} & 1\text{-cells} & 2\text{-cells} \\
\cdot & \circ & \cdot \\
\end{array}
\]

\[+1 - 0 + 1 = 2 = X(\circ\circ)\]

\[
\begin{array}{ccc}
\cdot & \circ & \infty \\
\end{array}
\]

\[+1 - 2 + 1 = 0 = X(\circ\circ)\]