A BRIEF ON THE HELIX OF FIFTHS

CARL MCTAGUE

1. Introduction

The *helix of fifths* is a model for musical pitch[1] possessing a number of unique (and useful!) features. Most importantly, it contains a countable infinity of pitch classes and comes very close to capturing the algebraic structure of western tonal language and notation. As an algorithmic compositional tool, it provides a simple solution to the problem of intuitive enharmonic representation in scores (e.g. should a pitch be notated B♭ or A♯ to make it seem more natural to a human performer?). Furthermore, it enables new tuning possibilities of classical structures within the realm of computer music and is extremely useful as a foundation for more sophisticated harmonic models. It is strongly inspired by ideas Eric Regener put forth in his under-appreciated *Musical Notation and Equal Temperament: A Formal Study* (1973).

2. The helix

Let $H := \mathbb{Z}$. Then H may be considered a model for pitch classes by virtue of the function $\mathrm{nota} : H \to \mathbb{Z}_7 \times \mathbb{Z}$,

$$\mathrm{nota}(n) := (n \bmod 7, n \operatorname{div} 7) = (n \bmod 7, \lfloor n/7 \rfloor)$$

which maps H to traditional western musical notation if we interpret the first component as a traditional note letter in circle-of-fifths ordering i.e. $0_7 \mapsto \mathrm{F}, 1_7 \mapsto \mathrm{C}, 2_7 \mapsto \mathrm{G}, \ldots, 6_7 \mapsto B$ and interpret the second component as the accidental in the following way: $0 \mapsto \natural$, while $1 \mapsto \sharp, 2 \mapsto \sharp\sharp, \ldots$ and $-1 \mapsto \flat, -2 \mapsto \flat\flat, \ldots$ etc. In this way, $\mathrm{nota}(0) = \mathrm{F}$ and $\mathrm{nota}[H]$ is essentially:[2]

$$\ldots, \underline{\mathrm{F}\flat\flat\flat, \ldots}, \underline{\mathrm{F}\flat\flat, \ldots}, \underline{\mathrm{F}\flat, \ldots, \mathrm{E}\flat, \mathrm{B}\flat}, \underline{\mathrm{F}, \mathrm{C}, \mathrm{G}, \mathrm{D}, \mathrm{A}, \mathrm{E}, \mathrm{B}}, \underline{\mathrm{F}\sharp, \mathrm{C}\sharp, \ldots, \mathrm{B}\sharp}, \underline{\ldots, \mathrm{B}\sharp\sharp}, \underline{\ldots, \mathrm{B}\sharp\sharp\sharp}, \ldots$$

Notice that H is quite unusual; it represents a countable infinity of pitch classes (as opposed to the 12 in $\mathbb{Z}_{12}$). This is achieved by dispensing with the notion of enharmonic equivalence. For example, in H, $-1 \neq 11$, i.e. B♭ ≠ A♯.

Geometrically, one can imagine H as a helix winding forever upwards with an increasing number of sharps and forever downwards with an increasing number of flats. If one were to project this helix downwards onto the plane perpendicular to its axis, collapsing H into enharmonic equivalence classes, one would obtain the conventional circle of fifths.[3]

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[1] I.e. a mathematical structure M for which there is a tuning function $\phi_M : M \to \mathbb{R}$ assigning to each of its elements a frequency. An established example is the equal-tempered $\mathbb{Z}_{12}$, familiar from serial composition.

[2] The underlining is merely meant to help illuminate the structure of H.

[3] provided the helix were wound with period 12; for other purposes it is useful to imagine it with period 7 as underlined above.

The elements of H represent pitch classes, but not specific pitches. For example, $0 \in H$ denotes the pitch class F, but not a specific pitch (i.e. a specific octave). For this, we consider a special subset $\overline{H} \subset H \times \mathbb{Z}$,

$$\overline{H} := \{(h,p) \in H \times \mathbb{Z} = \mathbb{Z}^2 : 4h - p \in 7\mathbb{Z}\}$$

where the second coordinate p encodes the staff position measured from the F above middle C (henceforth referred to as middle F). The algebraic condition which elements of $\overline{H}$ satisfy ensure that they don't refer to nonsense such as a G notated in the staff position reserved for F.[4] $\overline{H}$ is closed under component-wise integer addition, which corresponds to accidental-sensitive transposition. We call $\overline{H}$ simply *the helix* (although it's structure is actually not really a helix, more a regular lattice of points on the plane).

Note that we could just as well have defined $\overline{H}$ with an octave rather than a staff position component. Then there would have been no need for an algebraic condition, but a meaningful addition would have been much less convenient to define.

The helix has a *positive cone*[5]

$$P := \{(h,p) \in \overline{H} : p \geq 0\}$$

and a corresponding absolute value[6]

$$|(h,p)| := (h \cdot \text{sign}(p), |p|).$$

We can think of P as the collection of musical intervals and think of $|\overline{x} - \overline{y}| = |\overline{y} - \overline{x}|$ as the musical interval between two notes $\overline{x}$ and $\overline{y} \in \overline{H}$. Table 1 presents elements of P within the octave, i.e. $(h,p) \leq_P (0,7)$, along with their traditional names in music theory. Note that the helix has *two different tritones*, just as in musical notation.

3. Tuning the helix

Because the helix distinguishes between enharmonically equivalent pitches, it grants the luxury of tuning them differently. The most immediate idea is to tune the helix in ascending perfect $\frac{3}{2}$ ratios, sending the pitch class $h \in H$ to the family

[4]Note that the condition corresponds to the relationship satisfied by the three most fundamental numbers in Western harmony: $4+7 = 11$ or, more idiomatically, $V + 8^{ve} = 12$.

[5]A positive cone P is a distinguished subset of a linear space closed under addition and multiplication by positive scalars. With positive cones, we may generalize inequalities; we may consider an inequality $a \leq_P b$ to be the statement $b - a \in P$. For example, the standard inequalities in $\mathbb{R}$ may be realized via the positive cone $\mathbb{R}^+$ of positive real numbers.

[6]An absolute value on a linear space X with a positive cone P may be thought of as a mapping $f : X \to P$ with the properties:

(1) $a \leq_P f(a)$ (i.e. $f(a) - a \in P$)
(2) $f(-a) = f(a)$
(3) $f(\lambda \cdot b) = |\lambda| \cdot f(b)$ where $|\lambda|$ is absolute value of λ in the underlying field.
(4) $f(a+b) \leq_P f(a) + f(b)$

for all $a, b \in X$ and all scalars λ. Although $\overline{H}$ lacks a scalar multiplication, it inherits all other properties directly from the absolute value on $\mathbb{Z}$.

$(h,p) \in P$	*musical interval*
$(-6,4)$	'minor' tritone
$(-5,1)$	minor second
$(-4,5)$	minor sixth
$(-3,2)$	minor third
$(-2,6)$	minor seventh
$(-1,3)$	perfect fourth
$(0,0)$	unison
$(1,4)$	perfect fifth
$(2,1)$	major second
$(3,5)$	major sixth
$(4,2)$	major third
$(5,6)$	major seventh
$(6,3)$	'major' tritone

TABLE 1. *Some* elements of P within the octave (there are a countable number) and their corresponding names in traditional music theory.

of frequencies $2^{\mathbb{Z}}\left(\frac{3}{2}\right)^h$. This leads to the following tuning function for $\overline{H}$.

$$\phi_{\bar{H}}(h,p) := \phi_{\bar{H}}(0,0) \cdot 2^{\mathrm{oct}(p)} \cdot \sup_{n\in\mathbb{Z}}\{3^h/2^n < 2\},$$

where $\mathrm{oct}(p) := p \text{ div } 7$ denotes the signed number of octaves between staff position p and middle F. The sup construction is necessary to ensure that $\phi_{\bar{H}}$ sends pitches within the proper octave. It seems quite likely that other interesting tuning functions are possible, but I have not yet imagined or explored them.

4. An application: a simple model for jazz harmonies

The helix is quite useful as a foundation for more elaborate models of harmony. I present a very simple toy model for jazz harmonies which is defined independently of any particular pitch model, but for which the helix is very well suited.[7]

A *scale* is a monotone embedding of the integers into a pitch model. That is, a function $s : \mathbb{Z} \to M$ where $M = \overline{H}$, $\mathbb{Z}_{12}$ or any other pitch model and $a < b \Rightarrow \phi_M(s(a)) \le \phi_M(s(b))$.[8] If one were of a conservative mind-set, one might further insist that a scale exhibit finite, uniform octave periodicity, i.e. that there exist an $m_s \in \mathbb{N}$ such that $\phi_M(s(n+m_s)) = 2\cdot\phi_M(s(n))$ for all $n \in \mathbb{Z}$. But let's not do that, since it precludes so many unusually interesting possibilities!

A *harmony* is then a triple consisting of a scale, a quality and a root, $(s,q,r) \in (\mathbb{Z} \to M)\times\mathbb{Z}\times M$. A *quality* is simply an integer, and a *root* simply a point of M. The meaning of this triple comes from the function $\pi_{(s,q,r)} : \mathbb{N} \to \mathcal{P}(M)$,

$$\pi_{(s,q,r)}(n) := \{r + s(q+2i) - s(q) : i \in \{0,\ldots,n-1\}\}.$$

Then $\pi_{(s,q,r)}(3)$ is the triad generated by the harmony (s,q,r) and $\pi_{(s,q,r)}(4)$ its seventh chord etc. We can achieve 'diatonic' progressions and substitutions by

[7] This model was in fact employed in the composition of the piece $(ii \to V \to I)^7$; please visit `http://www.mctague.org/carl/` to hear a recording.

[8] Note that we are using the linear ordering of M induced by the range of the tuning function $\phi_M : M \to \mathbb{R}$.

shifting q and r in parallel according to s (see $\text{dia}\sigma_n$ below) and achieve other substitutions and quality changes by keeping r fixed, while altering q.

To be more concrete, we explicitly construct some common harmonies over $\overline{H}$. First, we construct the major diatonic scale, $d : \mathbb{Z} \to \overline{H}$,

$$d(n) := ([0, 2, 4, -1, 1, 3, 5]_{(n \bmod 7)}, n),$$

where $[a_0, \ldots, a_n]_i$ denotes a_i. Then the F major harmony is $T := (d, 0, (0, 0))$, its tonic triad $\pi_{(d,0,(0,0))}(3)$ and its dominant seventh chord $\pi_{(d,4,(1,4))}(4)$.

Straight-forward algebraic operators may be defined on harmonies to simplify these constructions. For example, if we define the diatonic shift,

$$\text{dia}\sigma_n(s, q, r) := (s, q + n, r + s(q + n) - s(q)).$$

then the preceding dominant seventh may be written as $\pi_{\text{dia}\sigma_4(T)}(4)$. Other useful operators are also possible and secondary embellishment is easily accomplished. Furthermore, other practical scales, such as the octatonic, may be used.

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