New Applications *of* Differential Geometry *to* Big Data

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The Euler characteristic

An integer associated to a space:

$$\chi(\text{polyhedron}) = \#\{\text{vertices}\} - \#\{\text{edges}\} + \#\{\text{faces}\}$$

This number is independent of triangulation!

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More generally, if a space X can be decomposed into a finite number of "open cells":

$$X = \bigsqcup_\alpha C_\alpha \qquad \text{then} \qquad \chi(X) = \sum_\alpha (-1)^{\dim(C_\alpha)}$$

This number is independent of cell decomposition, even invariant under continuous deformation (homeomorphism, and for compact spaces even homotopy equivalence).

Euler Calculus for simple functions

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The Euler integral of a "**simple function**" is easy to define:

$$\int \Big(\sum_{\text{finite}} a_i \, 1_{V_i} \Big) \mathrm{d}\chi = \sum_{\text{finite}} a_i \, \chi(V_i) \qquad a_i \in \mathbf{R}, \quad V_i \subset X$$

(Known as a "constructible function" in algebraic geometry.)

For simple functions, Euler integration extends to a functor

Multiplicativity $\chi(Y \times Z) = \chi(Y) \cdot \chi(Z)$ implies the Fubini theorem for simple functions:

$$\int_Y \left(\int_Z s \, \mathrm{d}\chi \right) \mathrm{d}\chi = \int_{Y \times Z} s \, \mathrm{d}\chi = \int_Z \left(\int_Y s \, \mathrm{d}\chi \right) \mathrm{d}\chi$$

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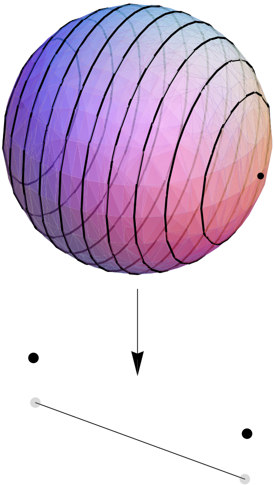
This pushforward f_* is **functorial** $(f \circ g)_* = f_* \circ g_*$.
If $c : X \to \mathrm{pt}$ then c_* is Euler integration.

Functoriality illustrated

Ex: $f : S^2 \to [-1, 1]$.

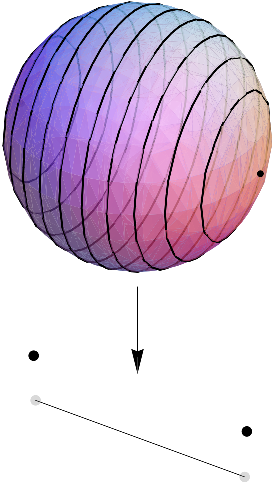


Functoriality illustrated

Ex: $f : S^2 \to [-1, 1]$.

generic fiber $\chi(S^1) = 0$

exceptional fiber $\chi(\mathrm{pt}) = 1$



$$2 = \chi(S^2) = \int 1_{S^2}\, d\chi = \int f_*(1_{S^2}) d\chi = \int (1_{\{1\}} + 1_{\{-1\}}) d\chi = 2$$

Functoriality in algebraic geometry

Riemann-Hurwitz formula: Applied to a ramified cover of Riemann surfaces $f : X \to Y$, functoriality gives:

$$\chi(X) = \deg(f) \cdot \chi(Y) - \sum_{x \in X} (e_x - 1)$$

where e_x is the ramification index of x.

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The **higher direct image** (Grothendieck 1952) $\mathrm{R}f_!$ lifts the pushforward f_* to sheaves: the Euler characteristics of the stalks of a sheaf $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(X)$ determine a simple function $\chi(\mathcal{F})$ and:

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Grothendieck-Deligne-MacPherson formalized Chern classes for singular varieties as a natural transformation $\mathrm{E} \to \mathrm{H}_*(-, \mathbf{Z})$. Functoriality lets one compare with smooth resolutions.

Functoriality is useful for data analysis

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Proof: Consider targets as tubes in spacetime. A tube has $\chi = 1$ so:

$$\#\{\textit{targets}\} = \int 1_{\text{tubes}}\, \mathrm{d}\chi = \int f_*(1_{\text{tubes}})\, \mathrm{d}\chi = \int (\text{local counts})\, \mathrm{d}\chi$$

where f : spacetime $\rightarrow$ space is the projection.

What about non-simple integrands?

Lebesgue integral is determined by its values on simple functions.
Eg, if a sequence of simple functions converges uniformly:

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As a result:

$\lim s_n = \lim s'_n$ doesn't necessarily imply that

$$\lim \int s_n \, \mathrm{d}\chi = \lim \int s'_n \, \mathrm{d}\chi$$

even if convergence is uniform.

The 2010 work of Baryshnikov & Ghrist

Baryshnikov-Ghrist studied this failure of convergence.

They considered the Euler integrals of two sequences of simple functions converging to a given continuous function α:

$$\int \alpha \lfloor \mathrm{d}\chi \rfloor = \lim_{n\to\infty} \frac{1}{n} \int \lfloor n\alpha \rfloor \, \mathrm{d}\chi \quad \int \alpha \lceil \mathrm{d}\chi \rceil = \lim_{n\to\infty} \frac{1}{n} \int \lceil n\alpha \rceil \, \mathrm{d}\chi$$

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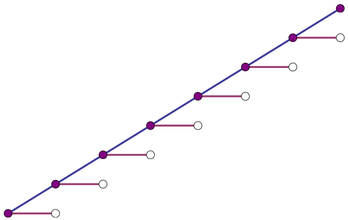
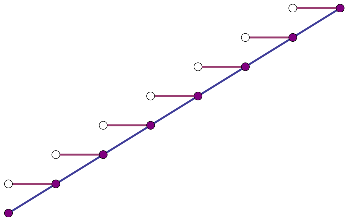
Ex:



$\int \mathrm{id}_{[0,1]} \lfloor d\chi \rfloor = 1$



$\int \mathrm{id}_{[0,1]} \lceil d\chi \rceil = 0$

The 2010 work of Baryshnikov & Ghrist

More generally:

Lemma (Baryshnikov-Ghrist): If $\alpha : \Delta^i \to \mathbf{R}$ is affine then:

$$\int_{\mathrm{int}(\Delta)} \alpha \lfloor \mathrm{d}\chi \rfloor = (-1)^i \inf \alpha \qquad \int_{\mathrm{int}(\Delta)} \alpha \lceil \mathrm{d}\chi \rceil = (-1)^i \sup \alpha$$

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Since inf and sup are not additive, neither of these integrals is.

Basic Question:

Can Euler integration be extended to continuous integrands in a way which is additive?

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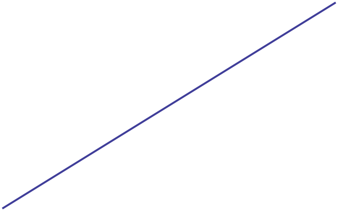
Ex: How to integrate $\mathrm{id}_{[0,1]}$?



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So for the time being:

- A **space** is a simplicial complex X.
- A **simple function** on X is an $\mathbf{R}$-linear combination of (the characteristic functions of) its simplices.
- A **continuous function** on X is a simplicial map $\alpha : X \to \mathbf{R}$ i.e. a function defined by assigning a real number to each vertex and extending linearly to the interior of each simplex.

In this context there is a *unique* simple function which best approximates a given continuous function $\alpha : X \to \mathbf{R}$, namely:

$$\sum_{\Delta \in X} \alpha(\hat{\Delta}) \cdot 1_{\mathrm{int}(\Delta)}$$

where $\hat{\Delta}$ is the barycenter of Δ.

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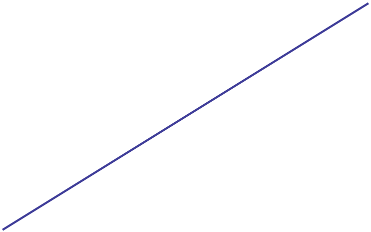

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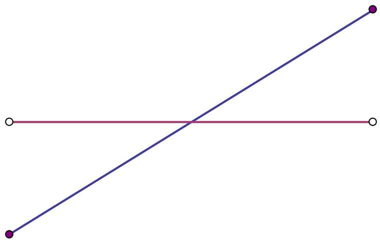

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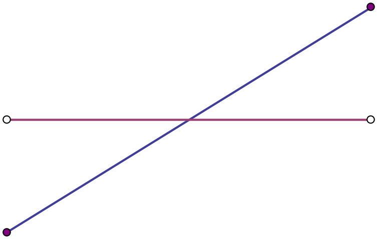

$$\int (1_{\{0\}} + 1_{\{1\}} + \tfrac{1}{2} \cdot 1_{(0,1)}) \mathrm{d}\chi = 0 + 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

Since $\chi(\mathrm{int}(\Delta)^i) = (-1)^i$ this approximation has Euler integral:

$$\sum_{\Delta^i \in X} (-1)^i \alpha(\hat{\Delta})$$

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Tentative Definition

For X and $\alpha : X \to \mathbf{R}$ simplicial *let:*

$$\int_X \alpha \, \mathrm{d}\chi = \sum_{\Delta^i \in X} (-1)^i \alpha(\hat{\Delta})$$

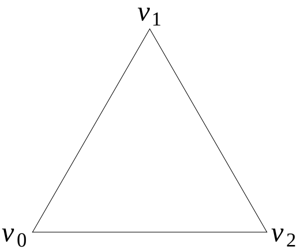
where the sum runs over each simplex Δ of X.

At the very least this integral is additive!

Exploration of the tentative definition's properties

It is *not* invariant under subdivision.

Ex:



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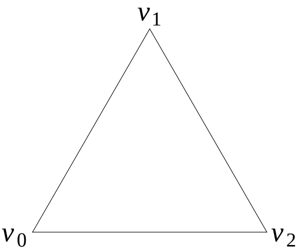
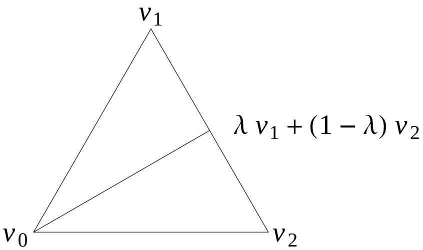




$$\int \alpha \, \mathrm{d}\chi = \alpha(\hat{\Delta}) = \tfrac{1}{3} \sum \alpha(v_i)$$

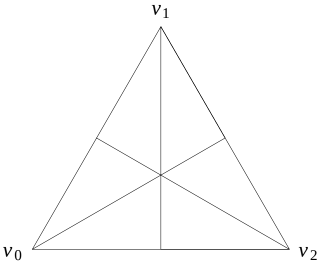
$$\int \alpha \, \mathrm{d}\chi = \tfrac{1}{6}\alpha(v_0) + \left(\tfrac{1}{3} + \tfrac{1}{6}\lambda\right) \alpha(v_1) + \left(\tfrac{1}{2} - \tfrac{1}{6}\lambda\right) \alpha(v_2)$$

These integrals differ for any $0 \leq \lambda \leq 1$.

(We shall return to this example later.)

Exploration of the tentative definition's properties, cont'd

But if one carries out a full barycentric subdivision then, after considerable calculation, one recovers the original integral.



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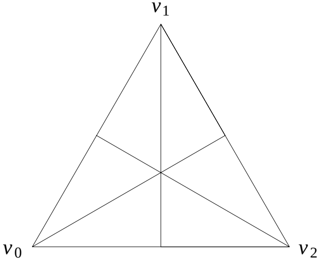

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Theorem: For any $n \geq 1$:

$$\int_X \alpha \; \mathrm{d}\chi = \int_{X^{(n)}} \alpha^{(n)} \; \mathrm{d}\chi$$

where $\alpha^{(n)} : X^{(n)} \to \mathbf{R}^{(n)}$ is the linear extension of α to the nth barycentric subdivision $X^{(n)}$ of X.

(This result appears in retrospect to have been a distraction though.)

Rewriting the sum

The tentative definition may be rewritten:

$$\int_X \alpha \, d\chi = \sum_{\Delta^i \in X} (-1)^i \alpha(\hat{\Delta})$$
$$= \sum_v \alpha(v) \mathrm{w}(v)$$

where v ranges over each vertex of X and where:

$$\mathrm{w}(v) = \sum_i (-1)^i \frac{1}{i+1} \#\{i\text{-simplices containing } v\}$$

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This number has a geometric interpretation!

Banchoff's 1967 work on curvature of embedded polyhedra

Let X be a simplicial complex *embedded in* $\mathbf{R}^n$.

Def (Banchoff): The *curvature* at a vertex v of X is:

$$\kappa(v) = \sum_{\Delta^i \in X} (-1)^i \mathcal{E}(\Delta^i, v)$$

where the excess angle $\mathcal{E}(\Delta^i, v)$ at v of a simplex $\Delta^i \subset \mathbf{R}^i$ is:

$$\mathcal{E}(\Delta^i, v) = \frac{1}{\mathrm{vol}(S^{i-1})} \int_{S^{i-1}} \left[\langle \xi, v \rangle \geq \langle \xi, x \rangle \text{for all } x \text{ in } \Delta^i \right] \mathrm{d}\xi$$

where ξ ranges over the unit sphere $S^{i-1} \subset \mathbf{R}^i$, and $[P] = \begin{cases} 1 & \text{if } P \\ 0 & \text{if } \neg P \end{cases}$

is the Iverson bracket.

Geometric interpretation of $w(v)$

Def: Given a simplicial complex X, let d_X be the intrinsic metric which makes each simplex flat and gives each 1-simplex length 1.

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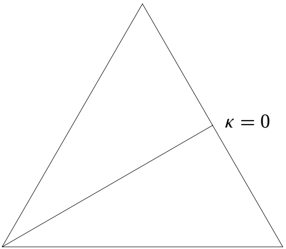
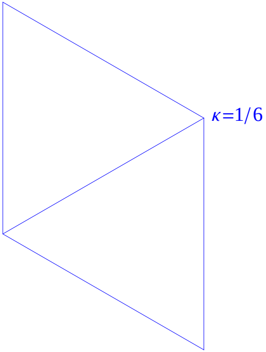
Ex: This explains why the integral isn't invariant under subdivision:



Should have integrated like this



but integrated like this instead.

Improved definition of integral

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Corrected Definition: For a metric simplicial complex X and a simplicial map $\alpha : X \to \mathbf{R}$, let:

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i.e. ***Euler integration is integration with respect to curvature***.

This makes a lot of sense actually...

Generalized Gauss-Bonnet theorem (1945)

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For a compact Riemannian manifold M:

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That is, **curvature is *infinitesimal Euler characteristic***.

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Simplicial Generalized Gauss-Bonnet Theorem (Banchoff):

$$\sum_v \kappa(v) = \chi(X)$$

*(Banchoff's work applies to **singular** simplicial complexes.)*

The importance of the boundary contribution

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Ex: An open interval $X = (0,1)$ has curvature 0 yet has $\chi(X) = -1$.
But if we write:

$$\int 1_{(0,1)} \, \mathrm{d}\chi = \int \left(1_{[0,1]} - 1_{\{0\}} - 1_{\{1\}}\right) \mathrm{d}\chi$$

then we can use curvature integration to correctly compute:

$$= (1/2 + 1/2) - 1 - 1 = -1$$

Curvature is as general as Euler characteristic

—i.e. it can be defined within any "O-minimal theory".

Bröcker-Kuppe's 2000 work on curvature of stratified spaces

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Loosely speaking, a Morse function $f : Y \to \mathbf{R}$ on a stratified space $Y \subset \mathbf{R}^N$ is one which restricts to a classical Morse function on each stratum.

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Definition (Goresky-MacPherson): The *local Morse data* at a critical point y of f is the pair:

$$(P, Q) = \mathrm{B}(y, \delta) \cap \left(f^{-1}[f(y) - \epsilon, f(y) + \epsilon],\, f^{-1}[f(y) - \epsilon] \right)$$

where $\mathrm{B}(y, \delta)$ is a closed ball of radius δ centered at y.

Bröcker-Kuppe's 2000 work on curvature of stratified spaces

Bröcker-Kuppe used Goresky-MacPherson's work on stratified Morse theory to define *curvature for any "tamely" stratified space.*

(This includes all spaces in an O-minimal theory.)

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Remark: P is always a cone so $\chi(P, Q) = \chi(P) - \chi(Q) = 1 - \chi(Q)$.

This number is called the *index* of f at y and denoted $\alpha(f, y)$.

Bröcker-Kuppe's 2000 work on curvature of stratified spaces

If Y is compact then:

$$\chi(Y) = \sum_{y \in Y} \alpha(f, y)$$

Definition (Bröcker-Kuppe): The **curvature measure** $\kappa_X(U)$ of a Borel set $U \subset X$ is:

$$\kappa_X(U) = \frac{1}{\mathrm{vol}(S^{N-1})} \int_{S^{N-1}} \sum_{y \in U} \alpha(f_x, y) \mathrm{d}x$$

where $f_x(y) = \langle x, y \rangle$ and x ranges over the unit sphere $S^{N-1} \subset \mathbf{R}^N$.

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Remark (Bröcker-Kuppe): If X is "tamely stratified" then f_x is a stratified Morse function for $\mathrm{d}S^{N-1}$ almost all x.

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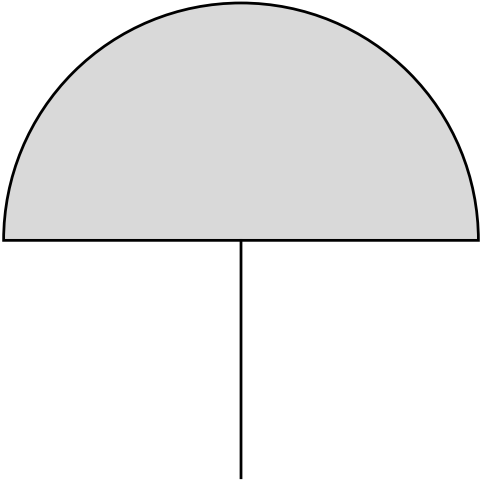
Remark: If X is a simplicial complex then the curvature measure is concentrated at the vertices, where it agrees with Banchoff's $\kappa(v)$.

Example from Bröcker & Kuppe's 2000 paper



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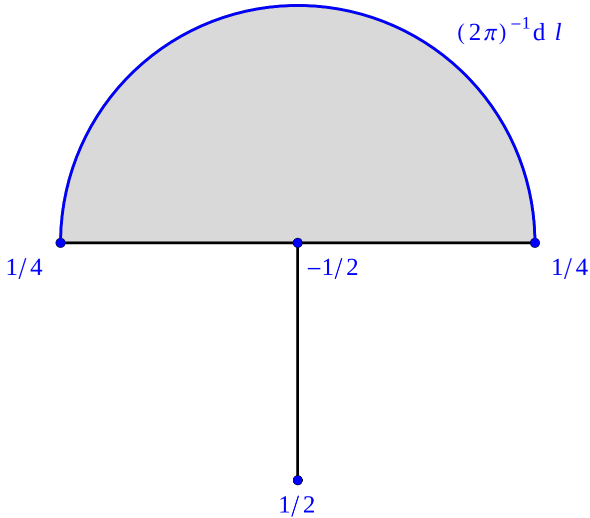

General Definition

Stratified Gauss-Bonnet Theorem (Bröcker-Kuppe):

If Y is compact then $\chi(Y) = \kappa_Y(Y)$, that is:

$$\int 1_Y \, \mathrm{d}\chi = \int 1_Y \, \mathrm{d}\kappa_Y$$

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So we reach:

Generalized Definition: For a compact tamely stratified space $Y \subset \mathbf{R}^N$ and a continuous function $\alpha : Y \to \mathbf{R}$, let:

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More generally, given continuous functions $\alpha_i : Z_i \to \mathbf{R}$ on compact tamely stratified subspaces $Z_i \subset Y \subset \mathbf{R}^N$, let:

$$\int_Y \sum_{\text{finite}} \alpha_i \, \mathrm{d}\chi = \sum_{\text{finite}} \int_{Z_i} \alpha_i \, \mathrm{d}\kappa_{Z_i}$$

Fubini Theorem

Since:

$$\kappa_{Y \times Z} = \kappa_Y \times \kappa_Z$$

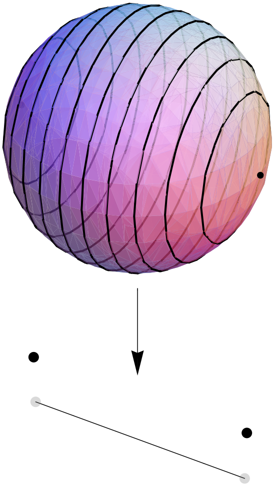
the **Fubini Theorem** holds:

$$\int_Y \left(\int_Z \alpha \, \mathrm{d}\kappa_Z \right) \mathrm{d}\kappa_Y = \int_{Y \times Z} \alpha \, \mathrm{d}\kappa_{Y \times Z} = \int_Z \left(\int_Y \alpha \, \mathrm{d}\kappa_Y \right) \mathrm{d}\kappa_Z$$

Basic Question:

Does curvature integration extend to a functor?

Ex: Revisiting the projection $S^2 \to [-1, 1]$



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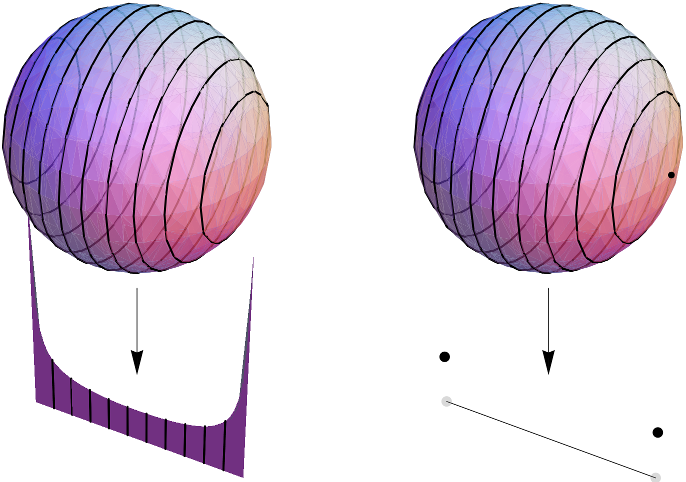
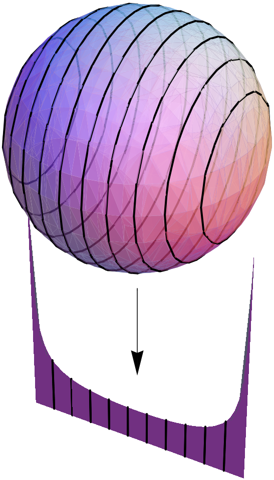
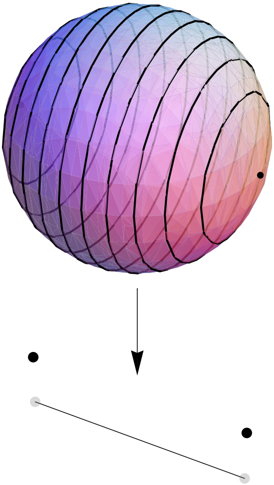


Ex: Revisiting the projection $S^2 \to [-1, 1]$





So although the classical pushforward depends only on the **intrinsic** geometry of the fibers, *the curvature pushforward depends also on the **extrinsic** geometry of the fibers!*

Karcher's formulation (1999) of the O'Neill formulas (1966)

A Riemannian submersion $f : M \to N$ splits $\mathrm{T}M$ into vertical and horizontal components:

$$\mathrm{T}M \cong \mathrm{V}M \oplus \mathrm{H}M$$

Let $\mathcal{H} : \mathrm{T}M \to \mathrm{T}M$ be the orthogonal projection onto $\mathrm{H}M$.

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Karcher's Formulas

If V is vertical and H horizontal then for any X, Y:

$$\begin{aligned} R(X,Y)V &= -\overbrace{R(X,Y)\mathcal{H}\cdot V}^{\textit{in }\mathrm{H}M} + \overbrace{R^{\mathrm{V}}(X,Y)V - \big[\nabla_X\mathcal{H}, \nabla_Y\mathcal{H}\big]V}^{\textit{in }\mathrm{V}M} \\ R(X,Y)H &= \underbrace{R(X,Y)\mathcal{H}\cdot H}_{\textit{in }\mathrm{V}M} + \underbrace{R^{\mathrm{H}}(X,Y)H - \big[\nabla_X\mathcal{H}, \nabla_Y\mathcal{H}\big]H}_{\textit{in }\mathrm{H}M} \end{aligned}$$

where $R^{\mathrm{H}}, R^{\mathrm{V}}$ are the curvatures of the induced connections on $\mathrm{H}M, \mathrm{V}M$.

Note: If X, Y are vertical then the second part of the second equation is the Gauss equation of the fibers.

The Pfaffian

The Generalized Gauss-Bonnet integrand is a certain multiple of the Pfaffian of the skew-symmetric matrix of 2-forms:

$$\left[\begin{array}{c|c} g\left(R(X,Y)V_i, V_j\right) & g\left(R(X,Y)H_i, V_j\right) \\ \hline g\left(R(X,Y)V_i, H_j\right) & g\left(R(X,Y)H_i, H_j\right) \end{array}\right] \mathrm{d}X\,\mathrm{d}Y$$

where $V_1, \ldots, V_k, H_{k+1}, \ldots, H_n$ is a basis for $\mathrm{T}M$ consisting of vertical and horizontal vectors.

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Karcher's formula lets us write:

$$\left[\begin{array}{c|c} g\left(\left(R^{\mathrm{V}}(X,Y) - [\nabla_X \mathcal{H}, \nabla_Y \mathcal{H}]\right)V_i, V_j\right) & -g\left(R(X,Y)\mathcal{H}\cdot H_i, V_j\right) \\ \hline g\left(R(X,Y)\mathcal{H}\cdot V_i, H_j\right) & g\left(\left(R^{\mathrm{H}}(X,Y) - [\nabla_X \mathcal{H}, \nabla_Y \mathcal{H}]\right)H_i, H_j\right) \end{array}\right] \mathrm{d}X\,\mathrm{d}Y$$

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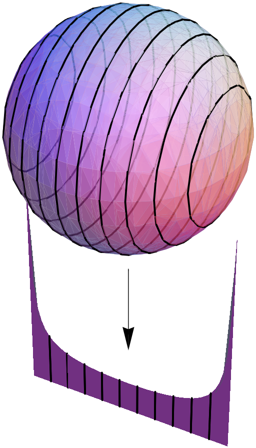
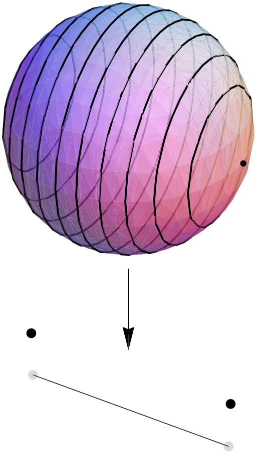
If the fibers are totally geodesic then $\nabla\mathcal{H} = 0$ and it reduces to:

$$\left[\begin{array}{c|c} g\left(\left(R^{\mathrm{V}}(X,Y)V_i, V_j\right) & 0 \\ \hline 0 & g\left(\left(R^{\mathrm{H}}(X,Y)H_i, H_j\right) \end{array}\right] \mathrm{d}X\,\mathrm{d}Y$$

so in this case the curvature splits $\mathrm{Pf}(\Omega_M) = \mathrm{Pf}(\Omega_N) \wedge \mathrm{Pf}(\Omega_F)$ and $f_*(\kappa_M) = \chi(F)\cdot\kappa_N$.

Classical pushforward as limit of curvature pushforward





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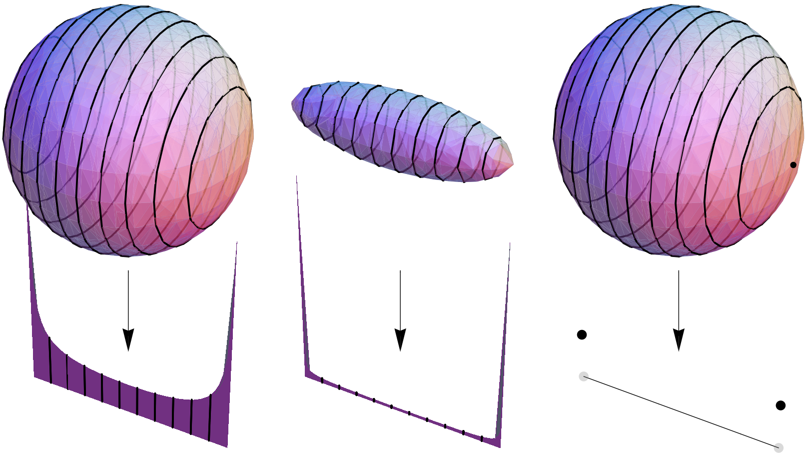

Shrinking the fiber ("Berger Deformation"): $g_\epsilon = g^{\mathrm{V}} + \epsilon \cdot g^{\mathrm{H}}$.

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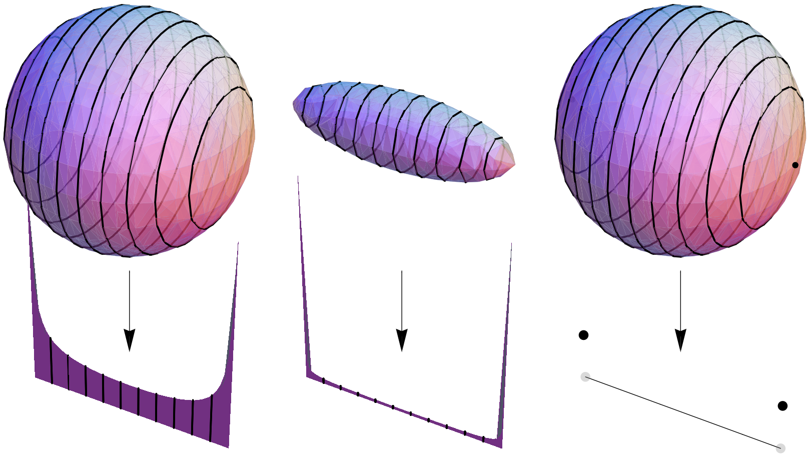

Shrinking the fiber ("Berger Deformation"): $g_\epsilon = g^{\mathrm{V}} + \epsilon \cdot g^{\mathrm{H}}$.

"Theorem": $f_*(\kappa_X^\epsilon) \to f_*(1_X) \cdot \kappa_Y$ as $\epsilon \to 0$.

Summary

Interpolating between Baryshnikov-Ghrist's non-additive:

$$\int_X \alpha \lfloor \mathrm{d}\chi \rfloor \qquad\qquad \int_X \alpha \lceil \mathrm{d}\chi \rceil$$

leads to an additive integral, and this integral is integration with respect to curvature:

$$\int_X \alpha \, \mathrm{d}\kappa_X$$

This integral is as general as the Euler characteristic itself.

It extends to a functor whose pushforward reflects both the intrinsic and *extrinsic* geometry of fibers.

This pushforward approaches the classical pushforward as one shrinks the fibers.

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