

# Music Through Computation

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International Mathematica Symposium

# Objective:

To develop powerful  
mathematical structures in order  
to *compose interesting new music*.

not to analyze existing music    although inspiration often  
comes from existing music and analytical techniques

Sound Spaces

Xi Operator

*6 Integers*  
2000

Helix of Fifths  
Model of Functional Harmony

*ii V I*<sup>7</sup>  
2002

Schenkerian Analysis

Dissonance Curves

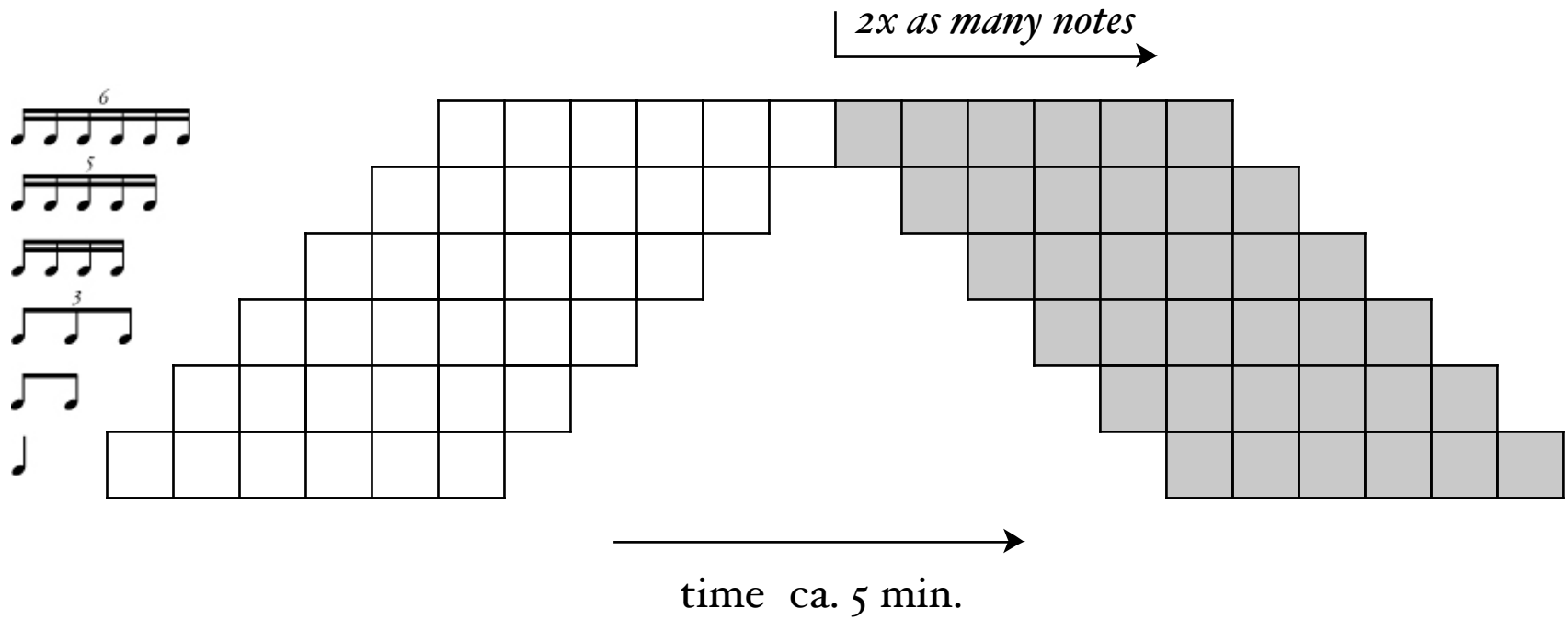
Schenkerian Synthesis

*Ripples Through Pitch Space*  
2003

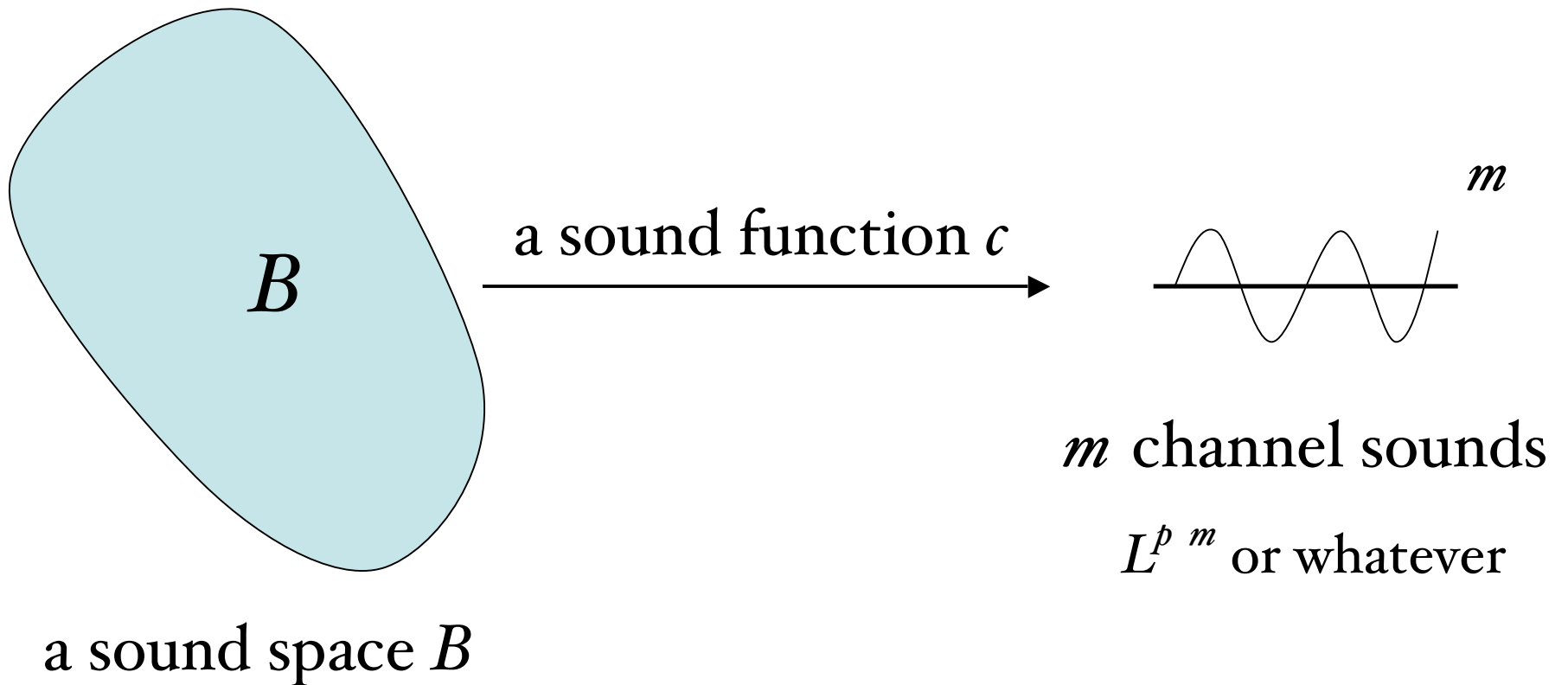
*Lament*  
work in progress

# 6 Integers

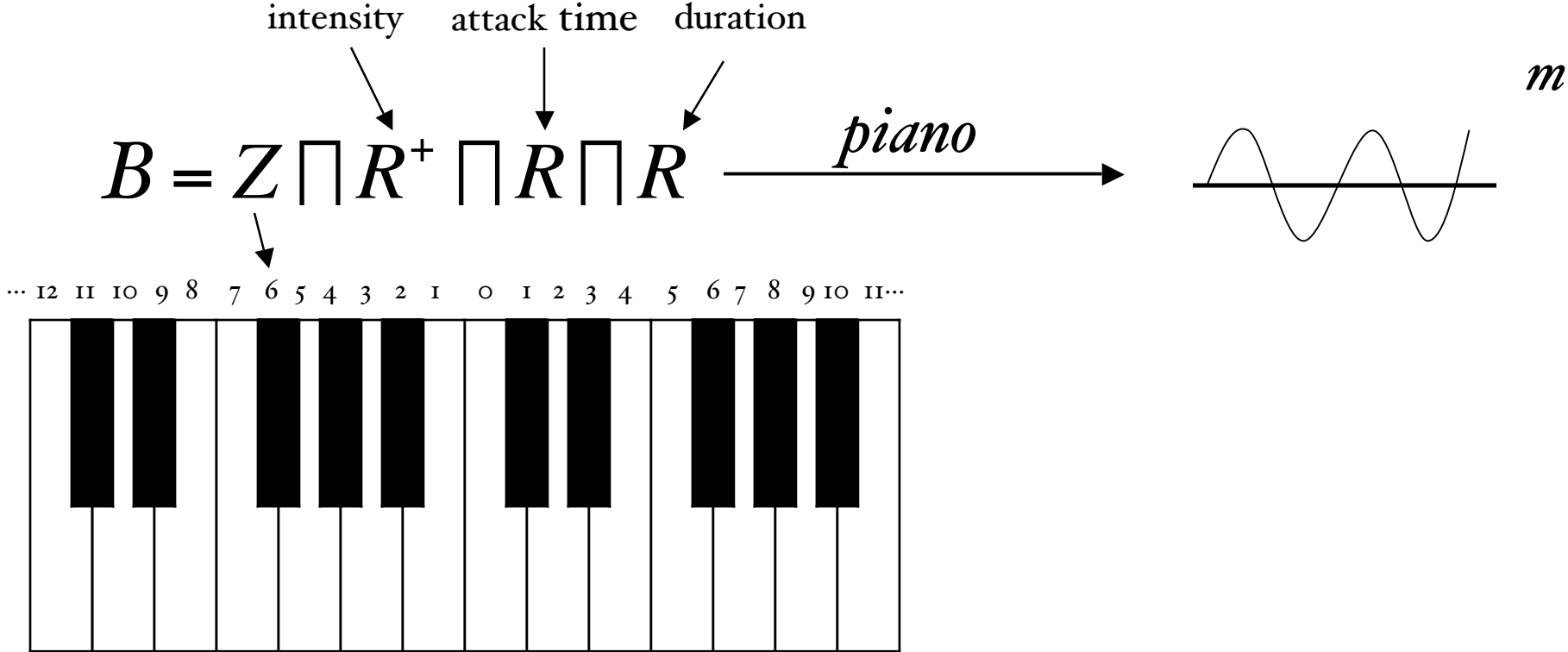
2000



# Sound Spaces



# Example: Piano



*But why bother with sound spaces at all?*

Why not just work directly  
within  $L^p$  or whatever ?

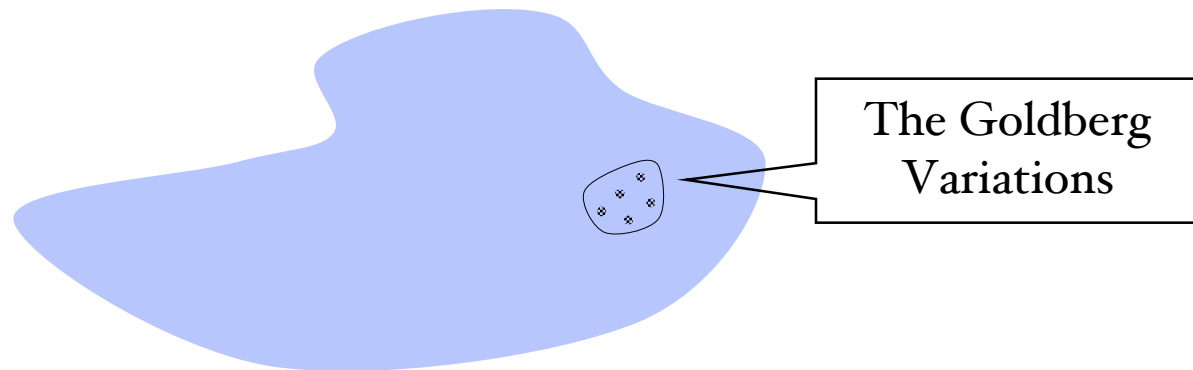
## *Why use sound spaces?*

- $L^p$  is dauntingly **HUGE!** Want to avoid the ultimate writer's block *how do you ever get started in the "space of all possible sounds"?*
- Want nice little representations of sounds inside the computer.

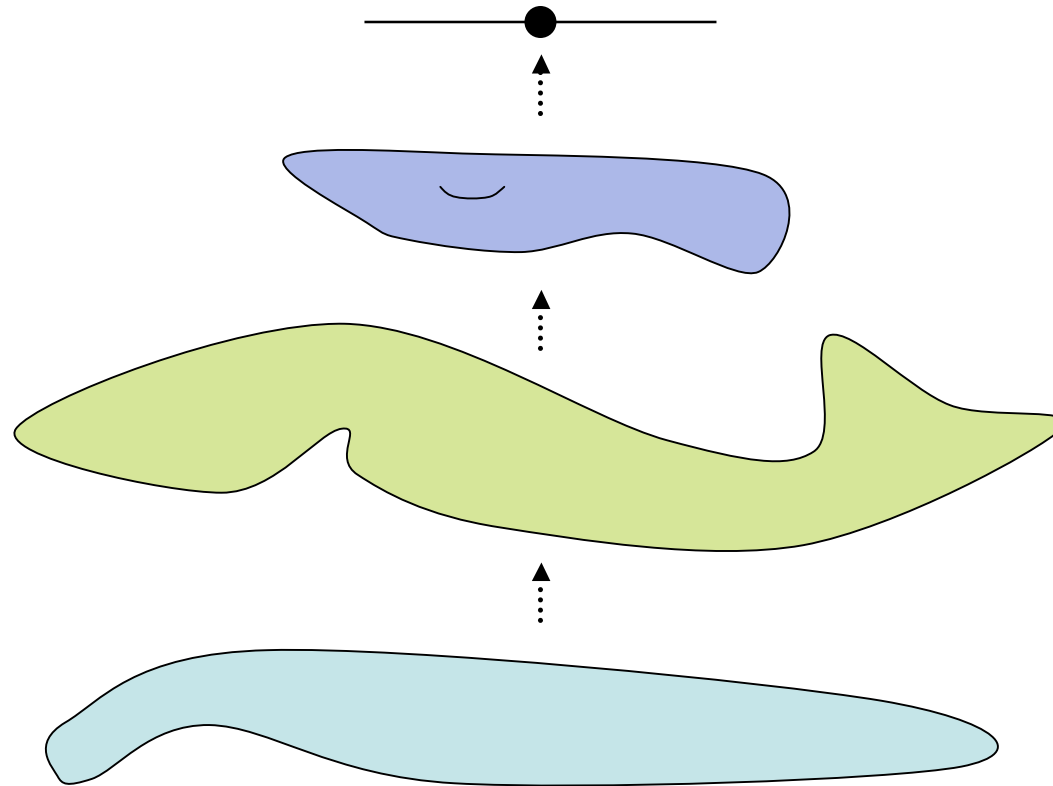


# *Why use sound spaces?*

- Want “musical topologies!” the standard metric on  $L^p$  is too rigid, unmusical.
- Natural e.g. continuous operations on the spaces should correspond to musical processes.
- E.g. variations might lie within neighborhoods:



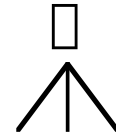
# General approach to composition



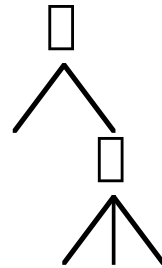
Inductively construct increasingly complex and specialized sound spaces until an entire piece of music is the image of a single, conspicuous point.

*Think of it as building increasingly powerful musical instruments.*

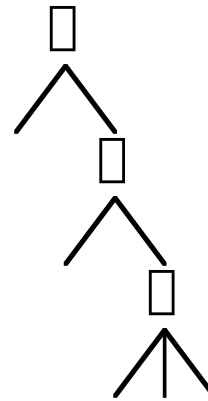
*Simple motivating example:*



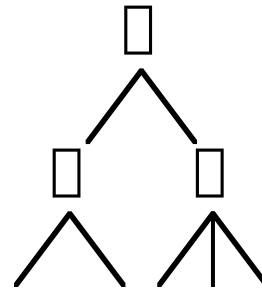
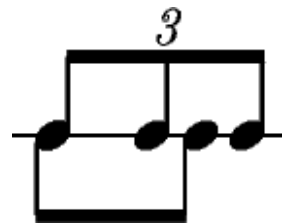
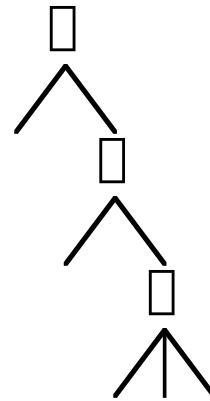
*Simple motivating example:*



*Simple motivating example:*



# *Simple motivating example:*

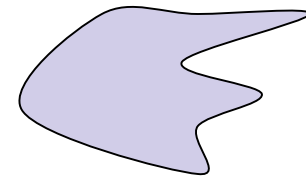
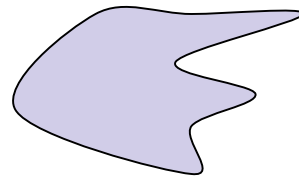
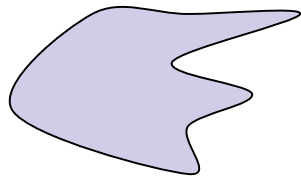


*the "Brahms rhythm"*

# *The General $X_i \boxplus$ Construction*

# *The General $X_i \sqcup$ Construction*

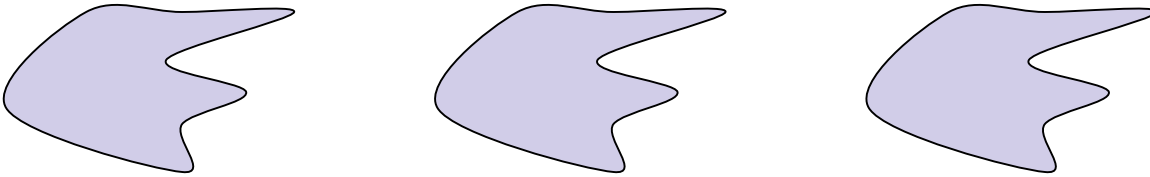
copies of an existing  
sound space  $B$   
each equipped with a  
potentially distinct sound  
function



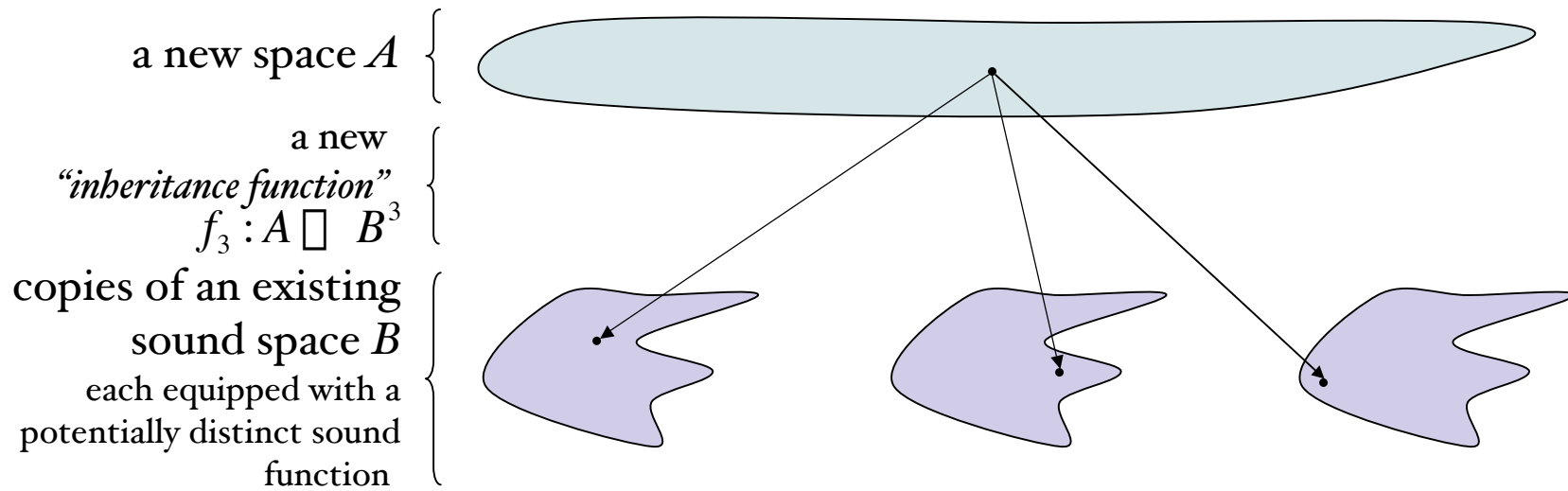


# *The General $X_i \sqcup B$ Construction*

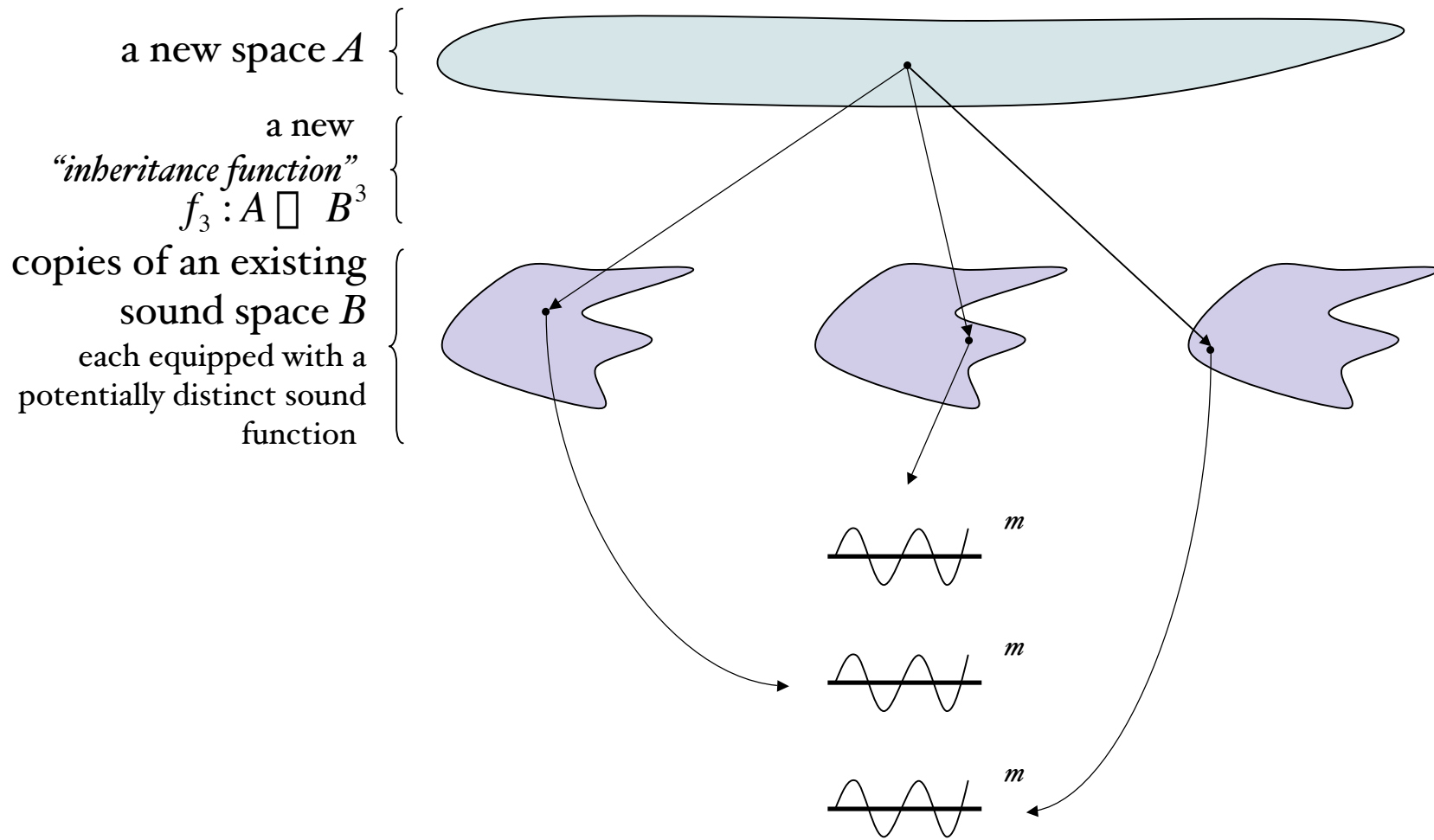
a new space  $A$  { 

copies of an existing  
sound space  $B$   
each equipped with a  
potentially distinct sound  
function { 

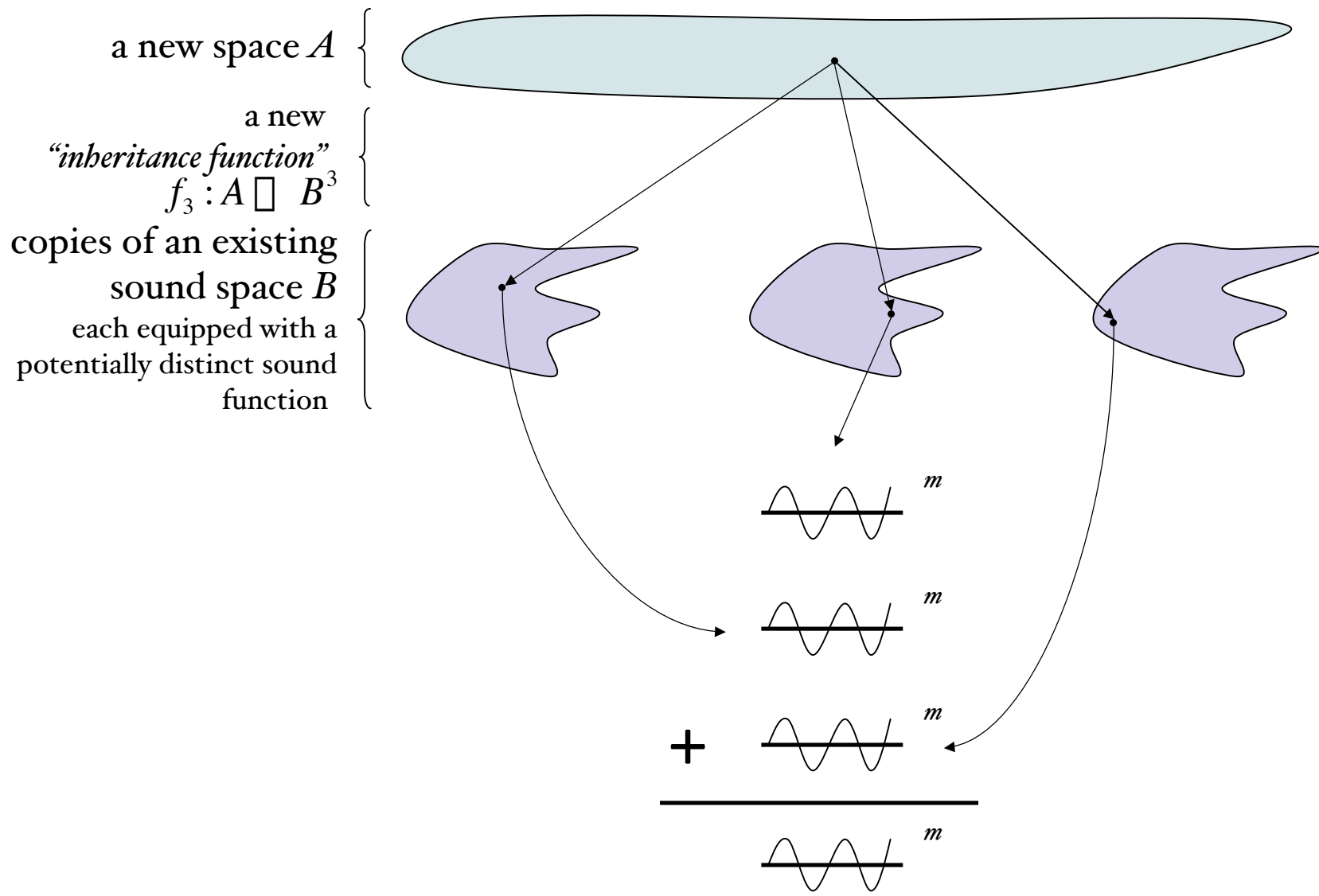
# The General $\Xi \boxplus$ Construction



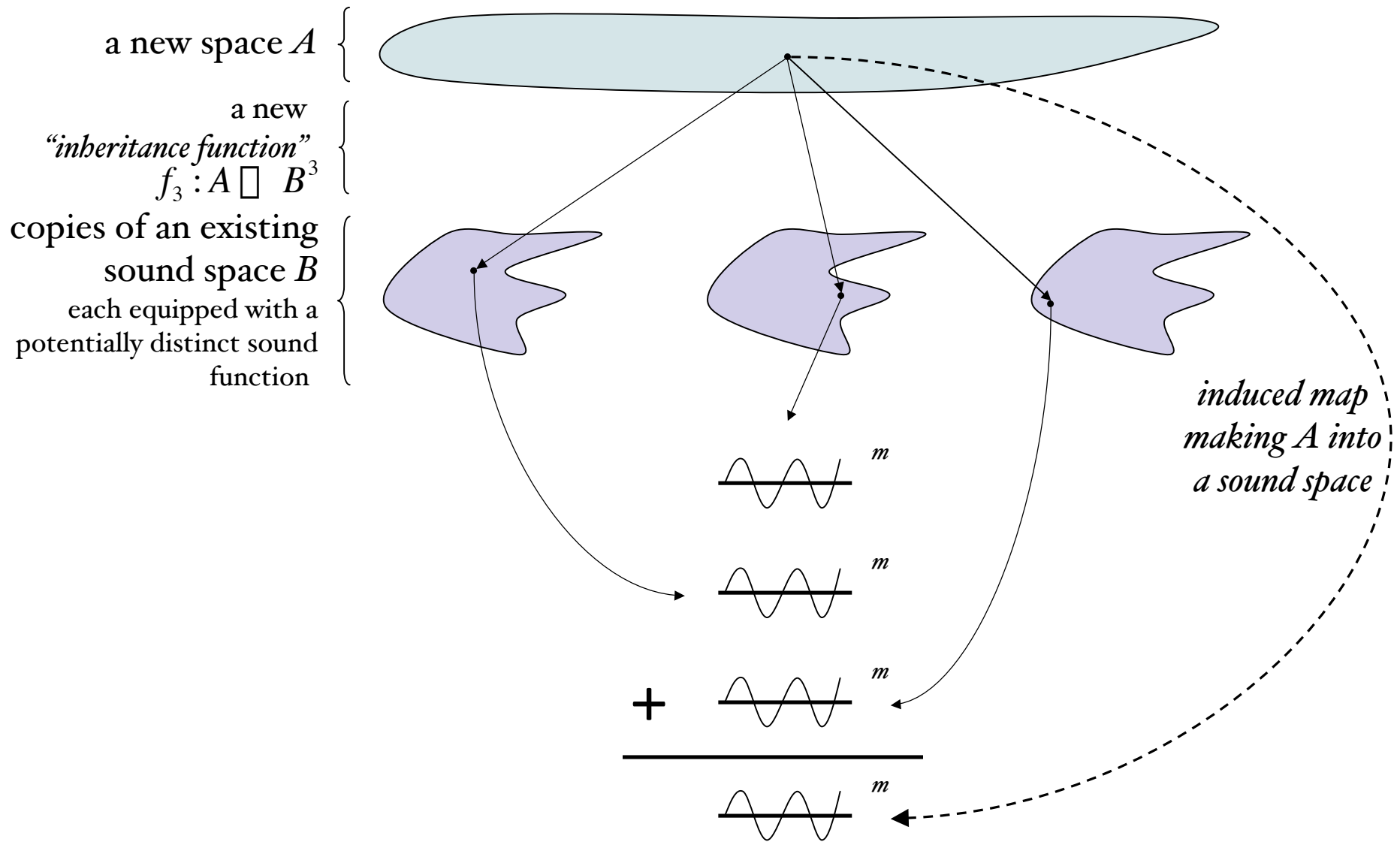
# The General $X_i \Xi$ Construction



# The General $\Xi \Xi$ Construction

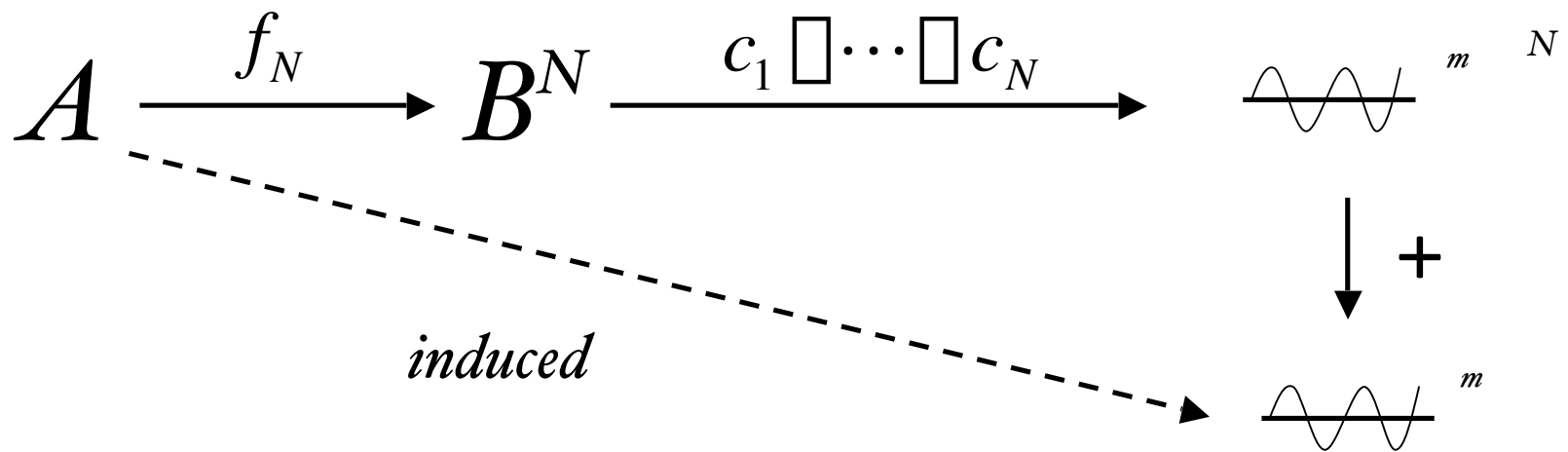


# The General $\Xi \boxplus$ Construction



# *Xi for diagram chasers:*

Given a list of sound functions  $c_i: B \rightarrow \text{[waveform]}^m$   $i=1, \dots, N$   
 and a family of “inheritance functions”  $f_n: A \rightarrow B^n$   
 make  $A$  into a sound space via the induced map:



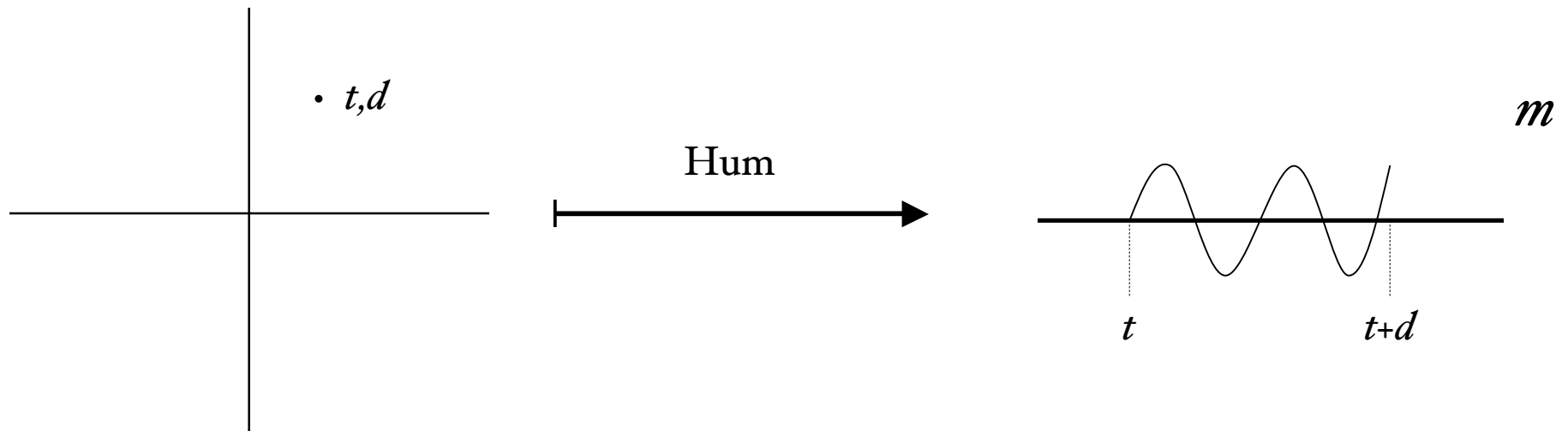
So, with  $\Xi$  in hand, we can build new sound spaces by constructing a few:

- fundamental sound spaces
  - families of inheritance functions
- and arranging them into hierarchies.

*This is precisely what we do, next...*

# A simple sound space:

Consider the plane  $\mathbb{R}^2$  as a sound space by regarding the point  $t, d$  as a hum at time  $t$  with duration  $d$ .



The so called time vector approach.



# Two Useful Families of Inheritance Functions:

The “diagonal maps”  
for simultaneity:

$$A \xrightarrow{\Delta_n} A^n$$

$$a \longmapsto a, \dots, a$$

just make  $n$  copies

For successiveness:

$$\mathbb{R}^2 \xrightarrow{\square_n} \mathbb{R}^{2n}$$


---


$$\downarrow \square_3$$


---


$$t, d \longmapsto t, d', t+d', d', \dots,$$

$$t + n \cdot d', d'$$

where  $d' = d/n$

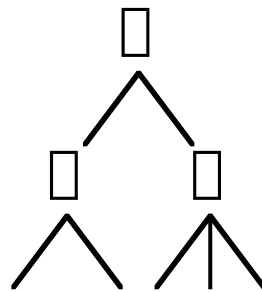
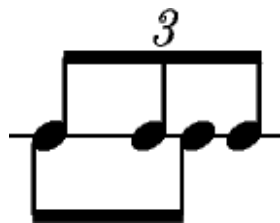
even subdivision of  
an interval into  $n$   
subintervals

# Application: *Rhythm Trees*



$$\Xi(\alpha, (H, \Xi(\alpha, (H, \Xi(\alpha, (H, H, H))))))$$

where H=Hum



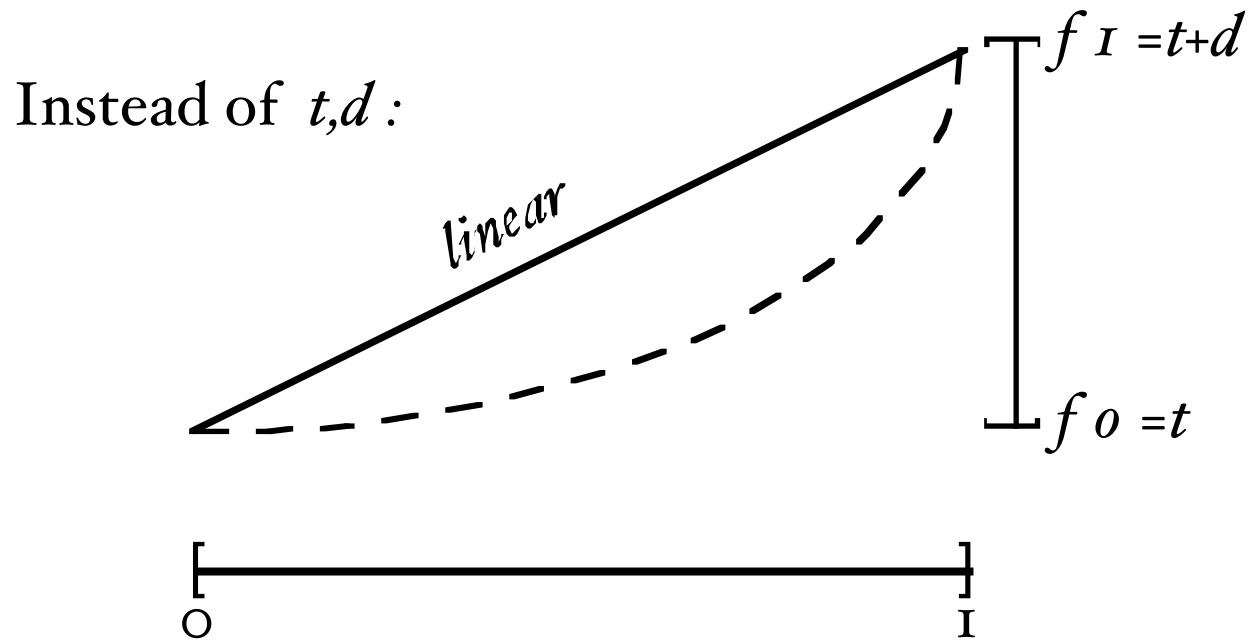
$$\Xi(\Delta, (\Xi(\alpha, (H, H)), \Xi(\alpha, (H, H, H))))$$

*the "Brahms rhythm"*

These functions, evaluated at  $o, I$  give the corresponding rhythms performed in the time interval  $o, I$ .

Instead of Time Vectors,  
Functions of the Unit Interval

$$O, I \rightarrow \mathbb{R}$$

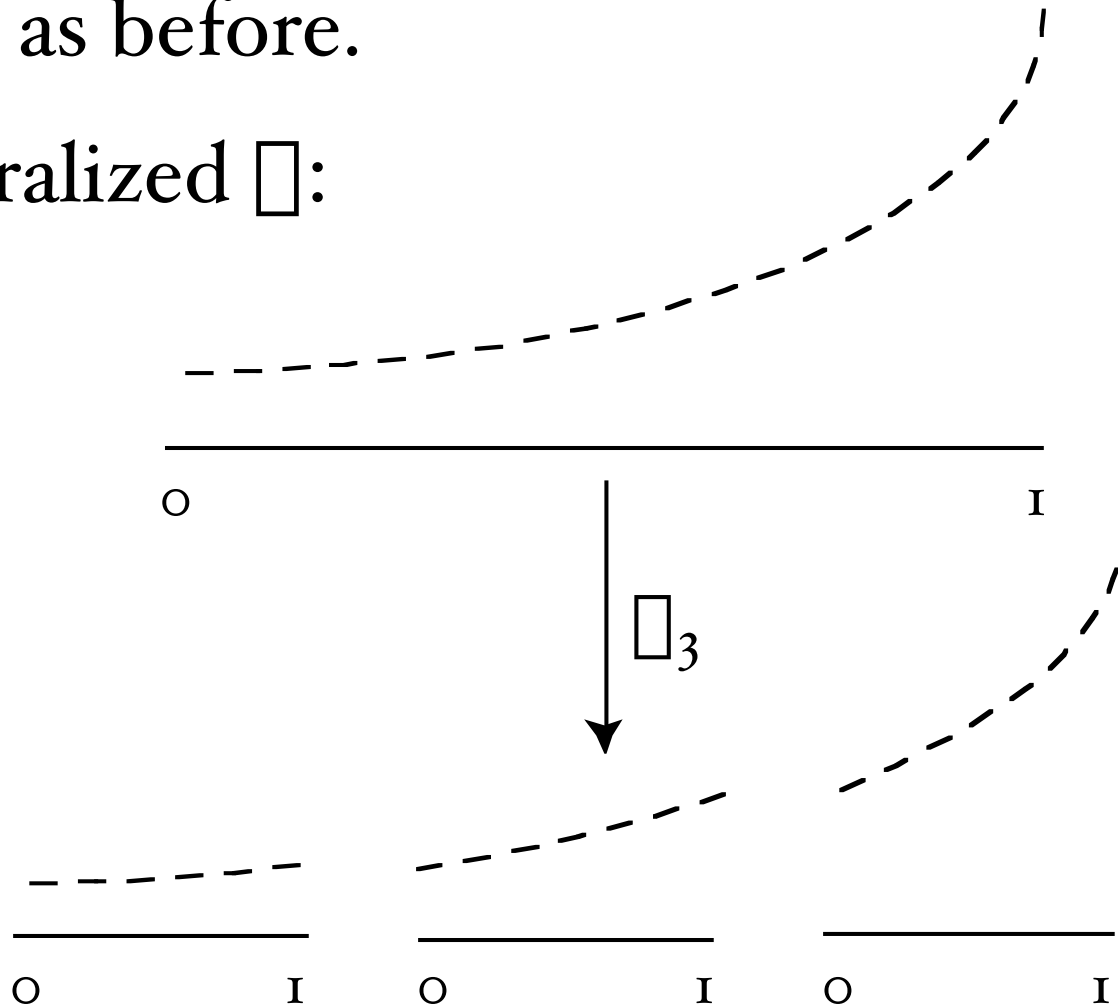


*But, we can use nonlinear functions to achieve accel and deceleration and expressive rhythms!*

# Inheritance Functions for $o, I \rightarrow \mathbb{R}$

$\square$  just as before.

Generalized  $\square$ :



*But why not just use time vectors and  
apply a global time map at the end?*

The hierarchical  $\mathcal{O}, \mathcal{I} \rightarrow \mathbb{R}$  approach  
permits local modification of the  
time map. Furthermore, different  
simultaneous components of a  
piece can have distinct time maps!

# Products of Inheritance Functions

We can form products of inheritance functions and thus pass several attributes of sound through the tree at once in parallel.

*E.g. rhythm, pitch, harmony,  
dynamics*

Sound Spaces

Xi Operator

*6 Integers*  
2000

Helix of Fifths

Model of Functional Harmony

*ii V I*<sup>7</sup>  
2002

Schenkerian Analysis

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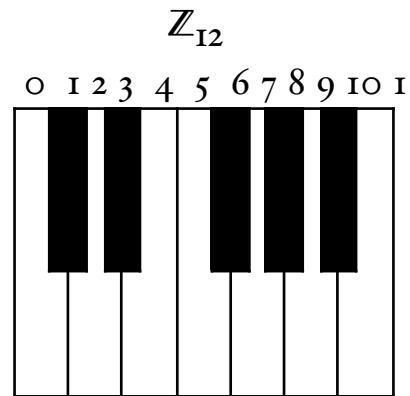
Schenkerian Synthesis

*Ripples Through Pitch Space*  
2003

*Lament*  
work in progress

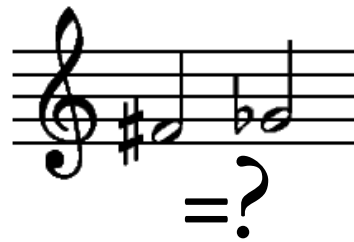


*“Circle of Fifths”*

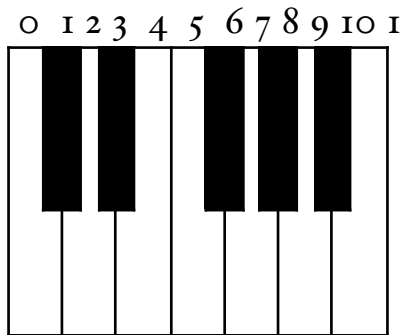




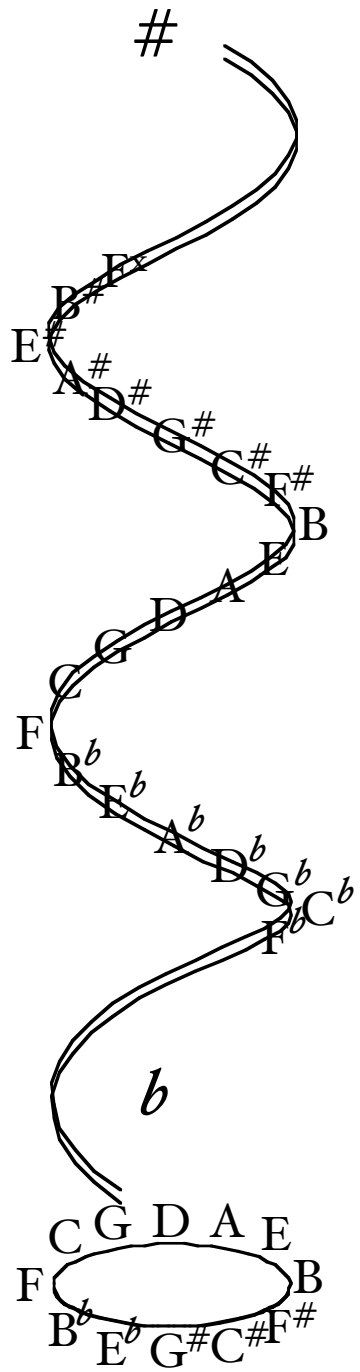
“Enharmonic equivalence”:



$Z_{12}$

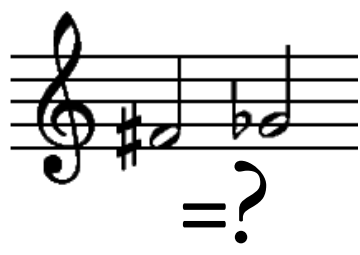


“Circle of Fifths”

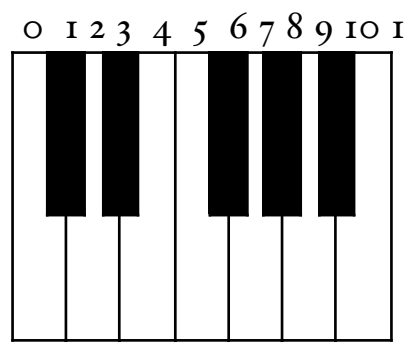


“Circle of Fifths”

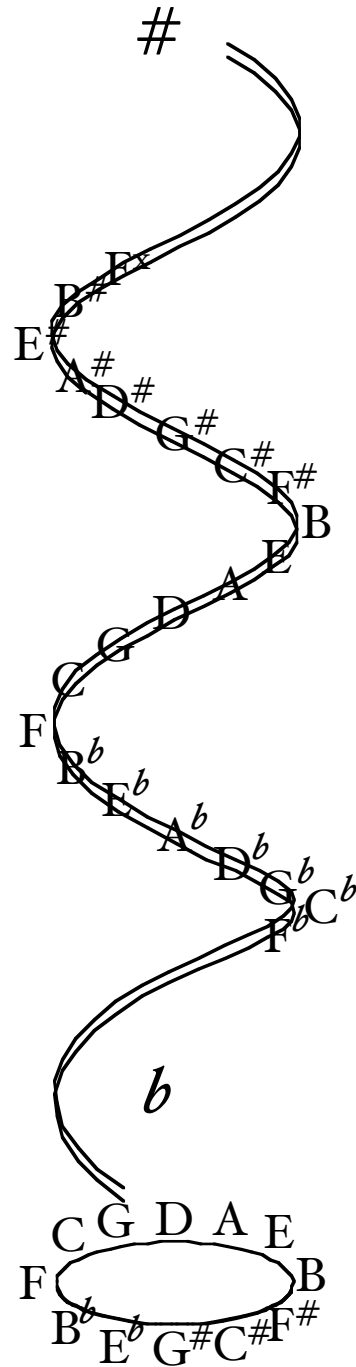
“Enharmonic equivalence”:



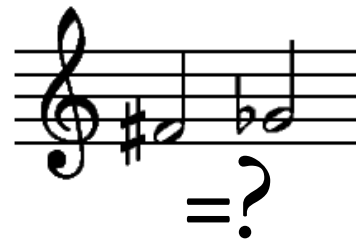
$Z_{12}$



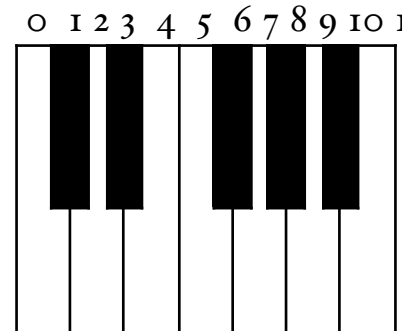
# Helix of Fifths



“Enharmonic equivalence”:



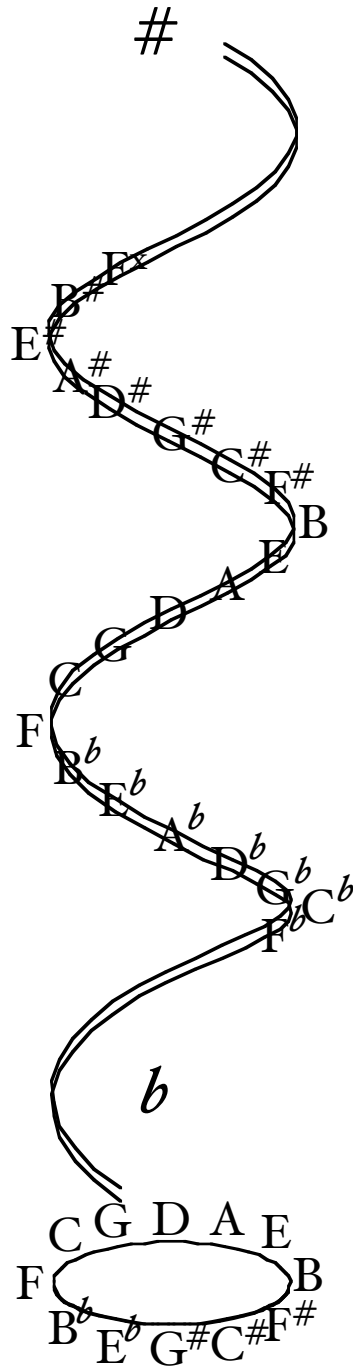
$Z_{12}$



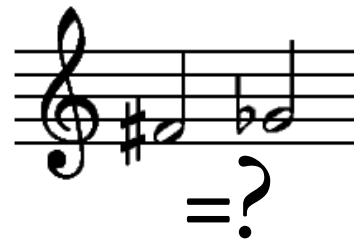
“Circle of Fifths”

# Helix of Fifths

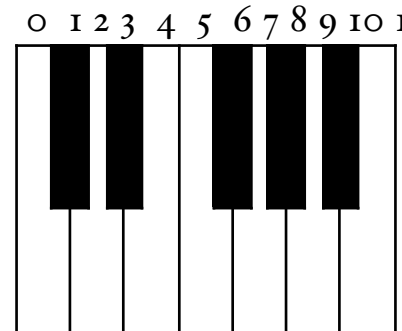
Strongly inspired by the work of Eric Regener



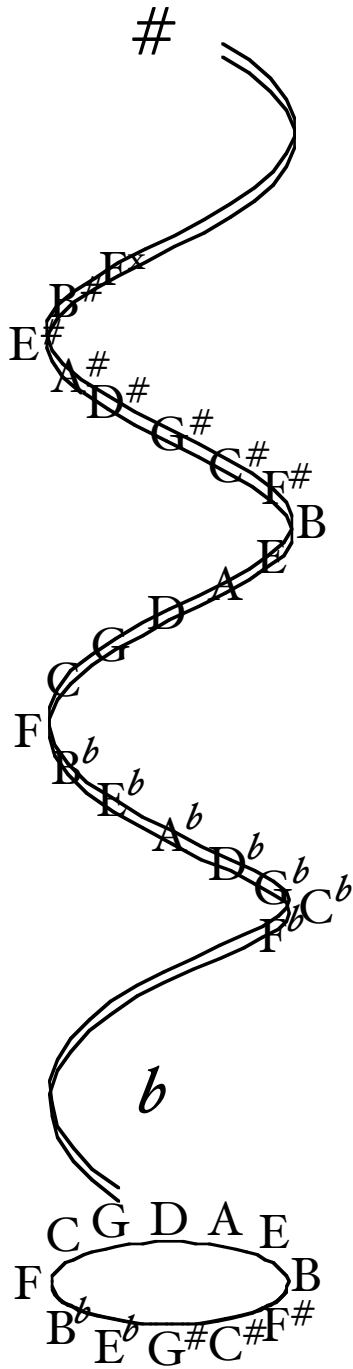
“Enharmonic equivalence”:



$Z_{12}$



“Circle of Fifths”



# Helix of Fifths

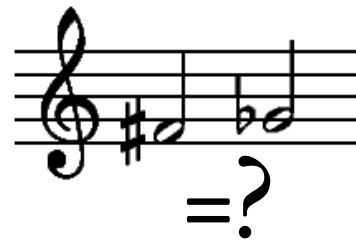
Strongly inspired by the work of Eric Regener

Give it the algebraic structure  $Z_7, +$ .

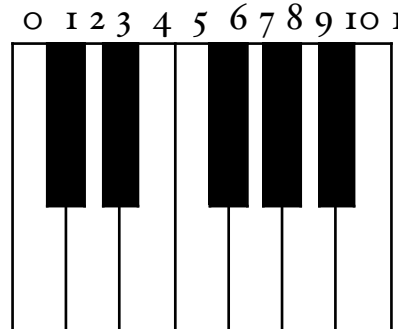
$$\text{nota}(n) := (n \bmod 7, \lfloor n/7 \rfloor)$$

*letter name, accidental*

“Enharmonic equivalence”:



$Z_{12}$



“Circle of Fifths”

Then, look at the sublattice:

$$\bar{H} := \{(h, p) \in \mathbb{Z}^2 : 4h \leq p \leq 7Z\} \subseteq (\mathbb{Z}^2, +)$$

where  $h$  is helix position and  $p$  is staff position.

It has a positive cone:

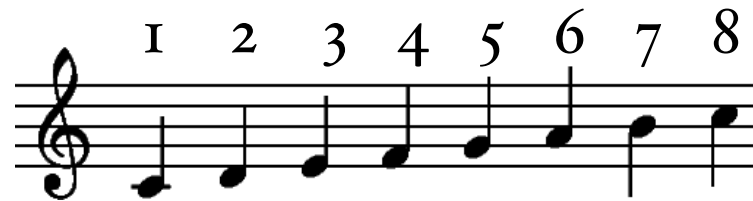
$$P := \{(h, p) \in \bar{H} : p \geq 0\}$$

and a corresponding absolute value:

$$|(h, p)| := (h \cdot \text{sign}(p), |p|).$$

# Brief Introduction to Functional Harmony

# Brief Introduction to Functional Harmony





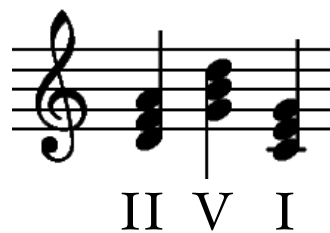
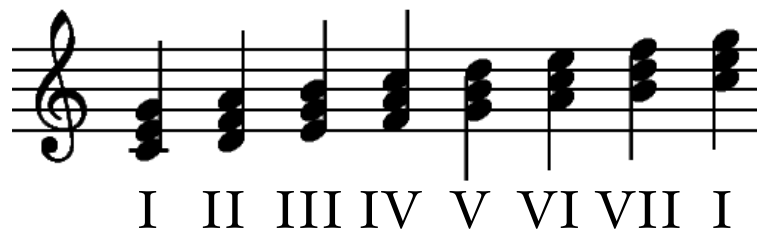
# Brief Introduction to Functional Harmony

The image displays two musical staves. The top staff shows a scale of eight notes: C, D, E, F, G, A, B, C. Above each note is a number from 1 to 8. The bottom staff shows the corresponding chords for each note: I, II, III, IV, V, VI, VII, I. The chords are represented by vertical lines with dots indicating the notes in the chord.

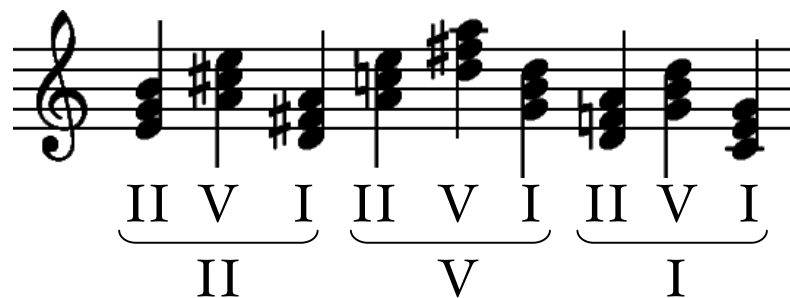
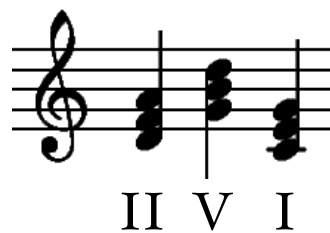
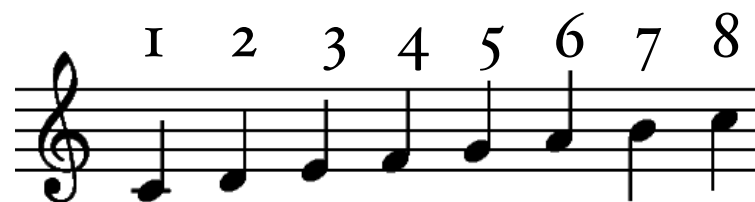
1 2 3 4 5 6 7 8

I II III IV V VI VII I

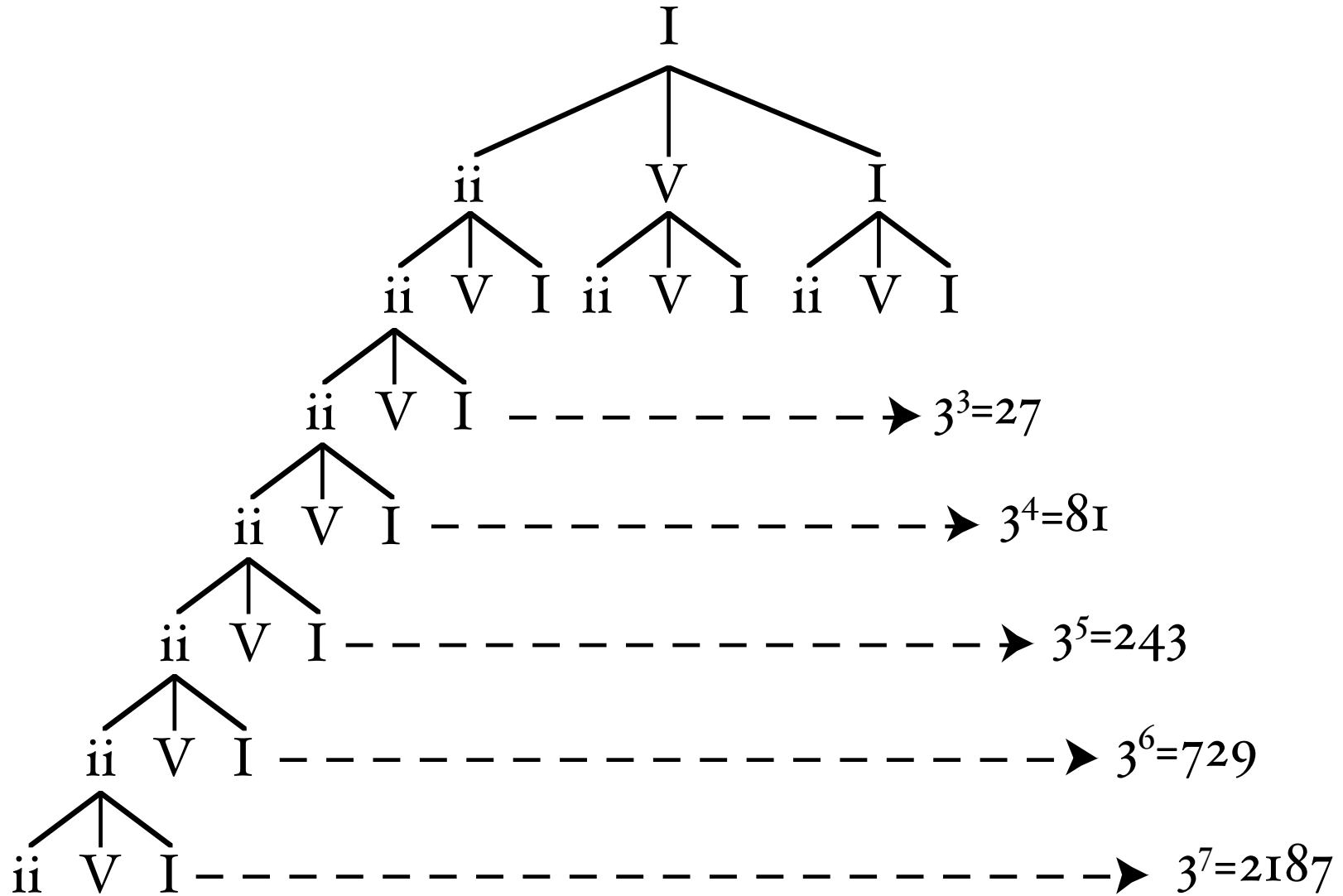
# Brief Introduction to Functional Harmony



# Brief Introduction to Functional Harmony



# ii V I <sup>7</sup> 2002



*total progression length: 3280*

$(ii-V-I)^7$ 

CARL MCTAGUE

*continuously accelerating*

12

20

26

32

39

46

8



74 <sup>8</sup>

77 <sup>8</sup>

80 <sup>8</sup>

83 <sup>8</sup>

86 <sup>8</sup>

89 <sup>8</sup>

92 <sup>8</sup>

95 <sup>8</sup>

Musical score for guitar, measures 98-119. The score is written in treble clef with a key signature of two sharps (F# and C#). The music consists of a series of chords and melodic lines, with a '8' above the first measure of each system, likely indicating an eighth note. The notation includes various chord voicings, including triads and dyads, and some melodic fragments. The piece concludes with a final chord in measure 119.

98

101

104

107

110

113

116

119



122 <sup>8</sup>

125 <sup>8</sup>

128 <sup>8</sup>

131 <sup>8</sup>

134 <sup>8</sup>

137 <sup>8</sup>

140 <sup>8</sup>

143 <sup>8</sup>

146 <sup>8</sup>

Musical staff 146: Treble clef, 8-measure rest, followed by a sequence of chords and eighth notes.

149 <sup>8</sup>

Musical staff 149: Treble clef, 8-measure rest, followed by a sequence of chords and eighth notes.

152 <sup>8</sup>

Musical staff 152: Treble clef, 8-measure rest, followed by a sequence of chords and eighth notes.

155 <sup>8</sup>

Musical staff 155: Treble clef, 8-measure rest, followed by a sequence of chords and eighth notes.

157 <sup>8</sup>

Musical staff 157: Treble clef, 8-measure rest, followed by a sequence of chords and eighth notes.

158 <sup>8</sup>

Musical staff 158: Treble clef, 8-measure rest, followed by a sequence of chords and eighth notes.

159 <sup>8</sup>

Musical staff 159: Treble clef, 8-measure rest, followed by a sequence of chords and eighth notes.

160 <sup>8</sup>

Musical staff 160: Treble clef, 8-measure rest, followed by a sequence of chords and eighth notes.

161 <sup>8</sup>

Musical staff 161, starting with a treble clef and a key signature of three sharps (F#, C#, G#). The staff contains a complex sequence of chords and melodic lines, including a prominent eighth-note pattern in the upper voice.

162 <sup>8</sup>

Musical staff 162, continuing the piece with similar harmonic and melodic structures as the previous staff.

163 <sup>8</sup>

Musical staff 163, featuring dense chordal textures and intricate melodic passages.

164 <sup>8</sup>

Musical staff 164, showing a continuation of the complex musical language with various rhythmic and harmonic elements.

165 <sup>8</sup>

Musical staff 165, maintaining the intricate texture of the previous staves.

166 <sup>8</sup>

Musical staff 166, with a focus on complex chordal relationships and melodic movement.

167 <sup>8</sup>

Musical staff 167, continuing the dense and detailed musical composition.

168 <sup>8</sup>

Musical staff 168, the final staff on this page, concluding the section with complex harmonic and melodic patterns.

169 <sup>8</sup>

170 <sup>8</sup>

171 <sup>8</sup>

172 <sup>8</sup>

173 <sup>8</sup>

174 <sup>8</sup>

175 <sup>8</sup>

176 <sup>8</sup>

177 <sup>8</sup>

178 <sup>8</sup>

179 <sup>8</sup>

180 <sup>8</sup>

181 <sup>8</sup>

182 <sup>8</sup>

183 <sup>8</sup>

184 <sup>8</sup>

185



Musical staff 185, featuring a treble clef, a key signature of three sharps (F#, C#, G#), and a common time signature. The staff contains a complex sequence of chords and melodic lines, including a prominent eighth-note pattern in the upper voice.

186



Musical staff 186, continuing the piece with a treble clef, three sharps key signature, and common time. The notation shows a continuation of the intricate harmonic and melodic textures from the previous staff.

187



Musical staff 187, featuring a treble clef, three sharps key signature, and common time. The staff contains a complex sequence of chords and melodic lines, including a prominent eighth-note pattern in the upper voice.

188



Musical staff 188, continuing the piece with a treble clef, three sharps key signature, and common time. The notation shows a continuation of the intricate harmonic and melodic textures from the previous staff.

189



Musical staff 189, featuring a treble clef, three sharps key signature, and common time. The staff contains a complex sequence of chords and melodic lines, including a prominent eighth-note pattern in the upper voice.

190



Musical staff 190, continuing the piece with a treble clef, three sharps key signature, and common time. The notation shows a continuation of the intricate harmonic and melodic textures from the previous staff.

191



Musical staff 191, featuring a treble clef, three sharps key signature, and common time. The staff contains a complex sequence of chords and melodic lines, including a prominent eighth-note pattern in the upper voice.

192



Musical staff 192, continuing the piece with a treble clef, three sharps key signature, and common time. The notation shows a continuation of the intricate harmonic and melodic textures from the previous staff.



201 <sup>8</sup>

202 <sup>8</sup>

203 <sup>8</sup>

204 <sup>8</sup>

205 <sup>8</sup>

206 <sup>8</sup>

207 <sup>8</sup>

208 <sup>8</sup>



209 <sup>8</sup>

210 <sup>8</sup>

211 <sup>8</sup>

212 <sup>8</sup>

213 <sup>8</sup>

214 <sup>8</sup>

215 <sup>8</sup>

216 <sup>8</sup>

The image displays a musical score for measures 209 through 216. Each measure is marked with a small '8' in a superscript, likely indicating an eighth note or eighth rest. The music is written in a treble clef with a key signature of three sharps (F#, C#, G#). The notation is highly complex, featuring dense chordal textures and intricate rhythmic patterns. The notes are often beamed together, and there are many accidentals throughout. The overall style is that of a highly technical and expressive piece of music.

217 <sup>8</sup>

218 <sup>8</sup>

219 <sup>8</sup>

220 <sup>8</sup>

221 <sup>8</sup>

222 <sup>8</sup>

223 <sup>8</sup>

224 <sup>8</sup>

225 <sup>8</sup>

Musical staff 225: Treble clef, key signature of two sharps (F# and C#), 8-measure rest. The staff contains a sequence of chords and melodic fragments, including a prominent F#5 chord in the first measure.

226 <sup>8</sup>

Musical staff 226: Treble clef, key signature of two sharps, 8-measure rest. Continuation of the musical sequence from staff 225.

227 <sup>8</sup>

Musical staff 227: Treble clef, key signature of two sharps, 8-measure rest. Continuation of the musical sequence.

228 <sup>8</sup>

Musical staff 228: Treble clef, key signature of two sharps, 8-measure rest. Continuation of the musical sequence.

229 <sup>8</sup>

Musical staff 229: Treble clef, key signature of two sharps, 8-measure rest. Continuation of the musical sequence.

230 <sup>8</sup>

Musical staff 230: Treble clef, key signature of two sharps, 8-measure rest. Continuation of the musical sequence.

231 <sup>8</sup>

Musical staff 231: Treble clef, key signature of two sharps, 8-measure rest. Continuation of the musical sequence.

232 <sup>8</sup>

Musical staff 232: Treble clef, key signature of two sharps, 8-measure rest. Continuation of the musical sequence.

233 <sup>8</sup>

234 <sup>8</sup>

235 <sup>8</sup>

236 <sup>8</sup>

*a tempo*

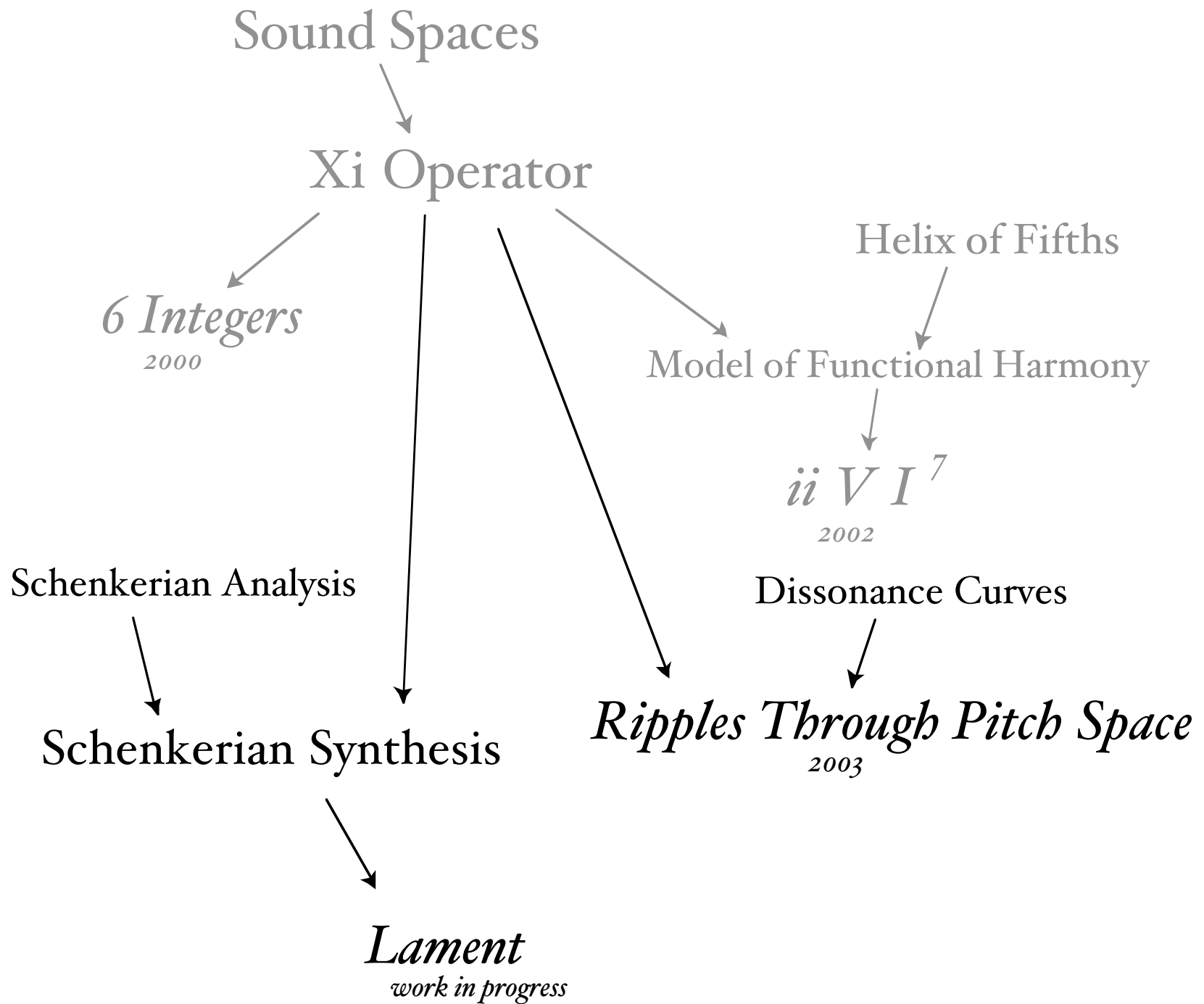
Here, the helix was fitted with  
“3 limit tuning.”

More generally: use factorization of  
rationals to biject  $\mathbb{Q}$  into  $\mathbb{Z}^{\infty} \hookrightarrow \mathbb{R}^{\infty}$ :  
finite support

$$\langle a_n \rangle \mapsto \prod p_i^{a_i}$$

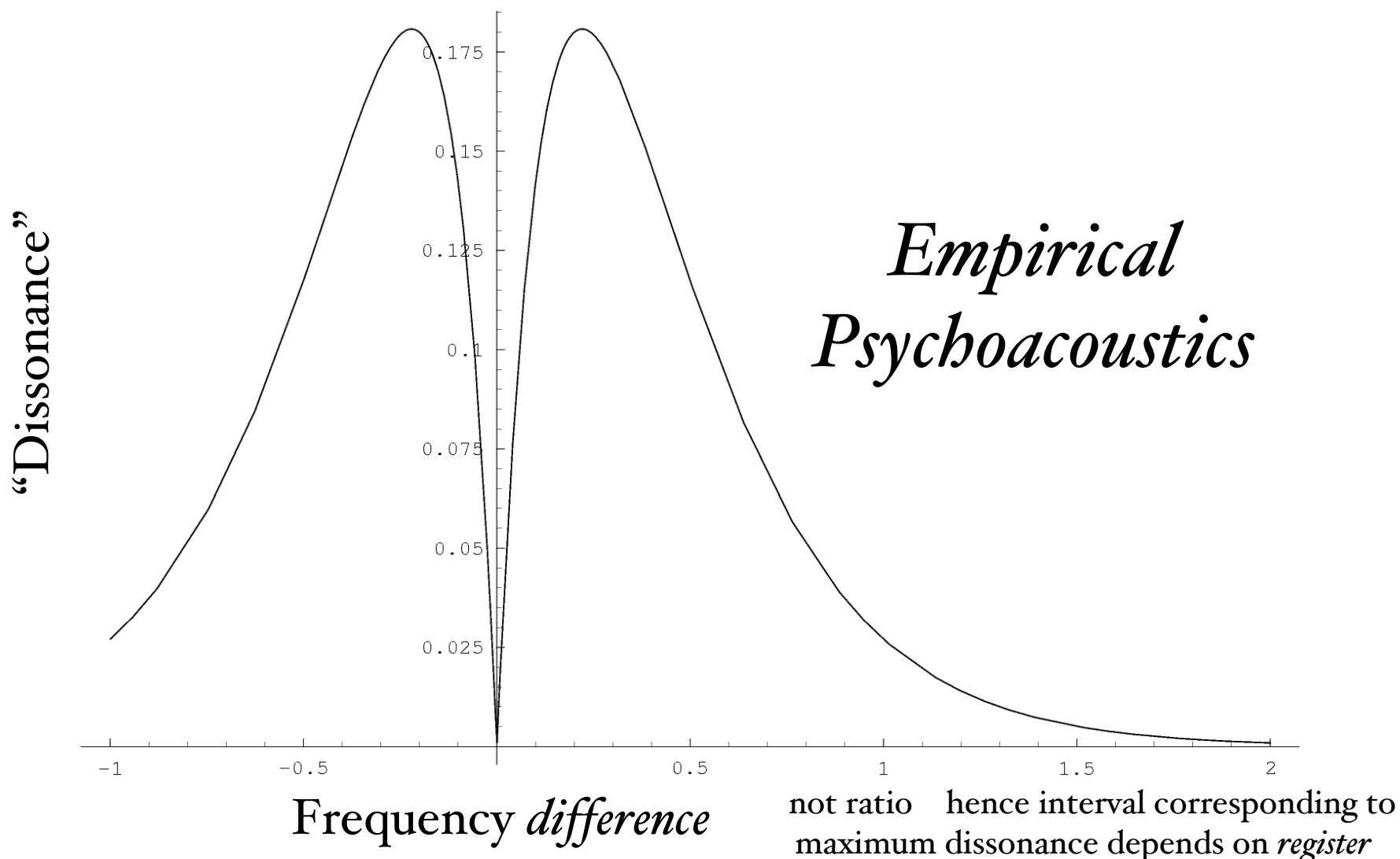
$p_i$  the  $i^{\text{th}}$  prime

and lift a nice metric from  $\mathbb{R}^{\infty}$ .

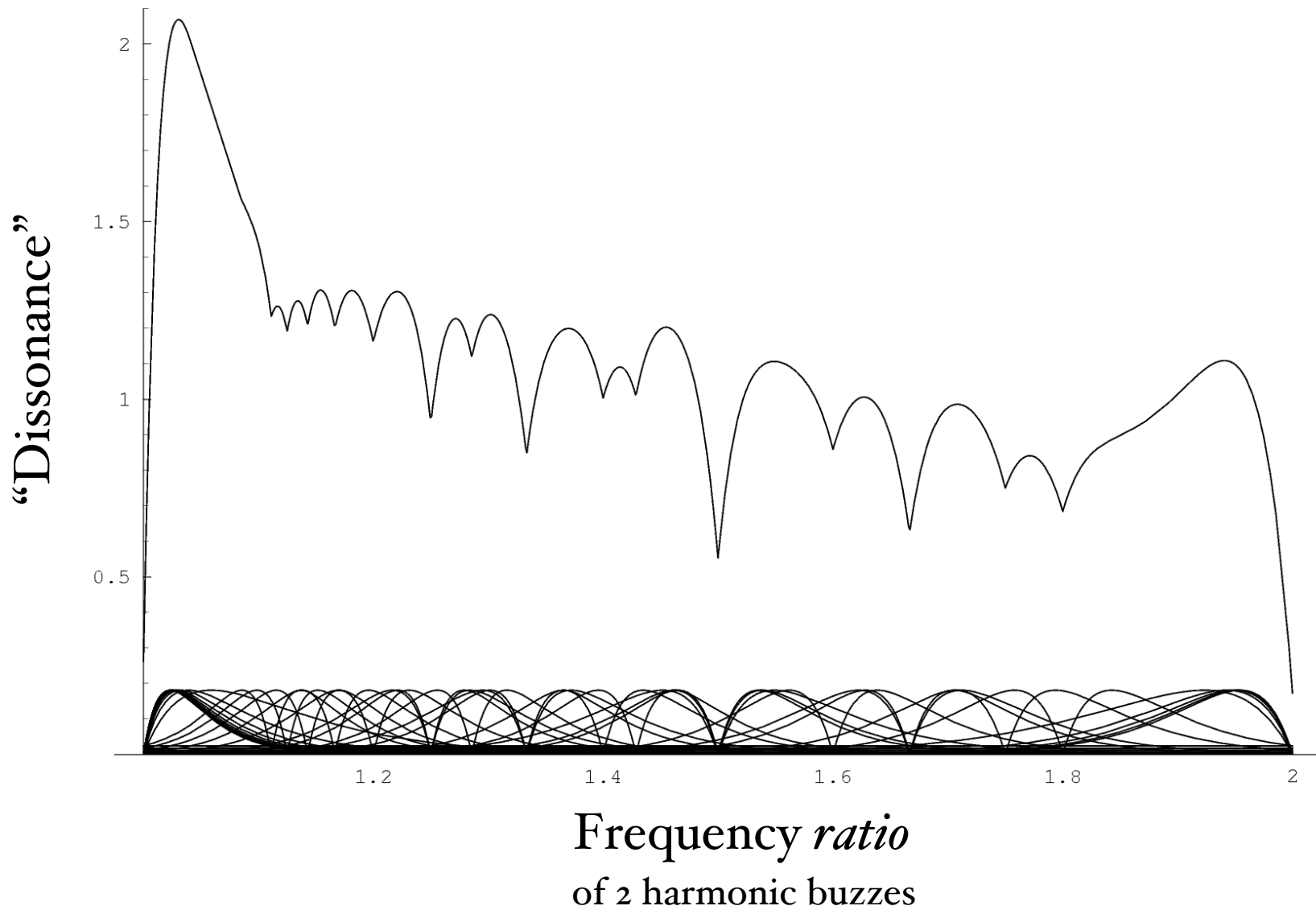


# Dissonance of 2 *Pure Sine Tones*

Kameoka, Kuriyagawa & Sethares



# Dissonance of 2 *Harmonic* “Buzzes”



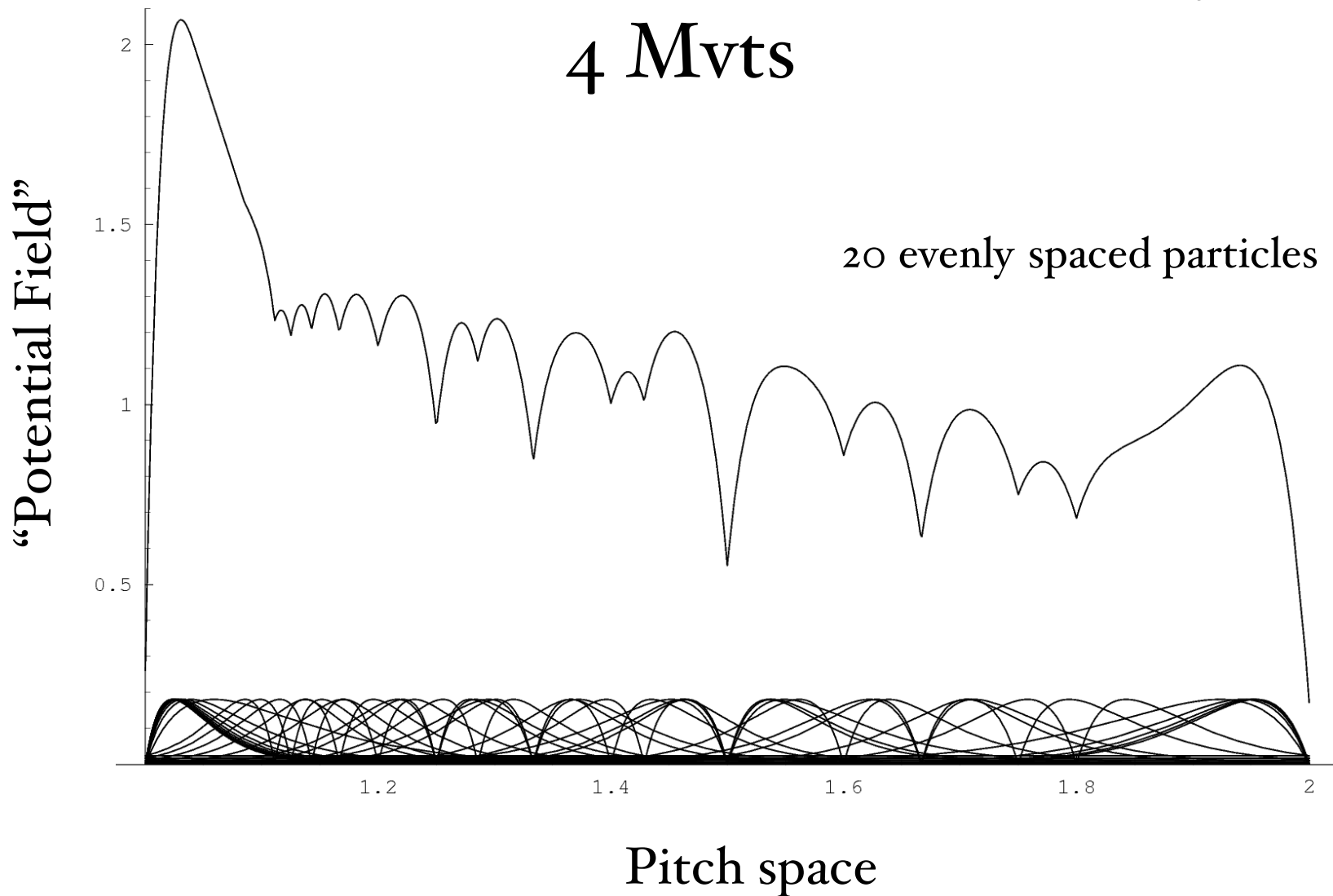


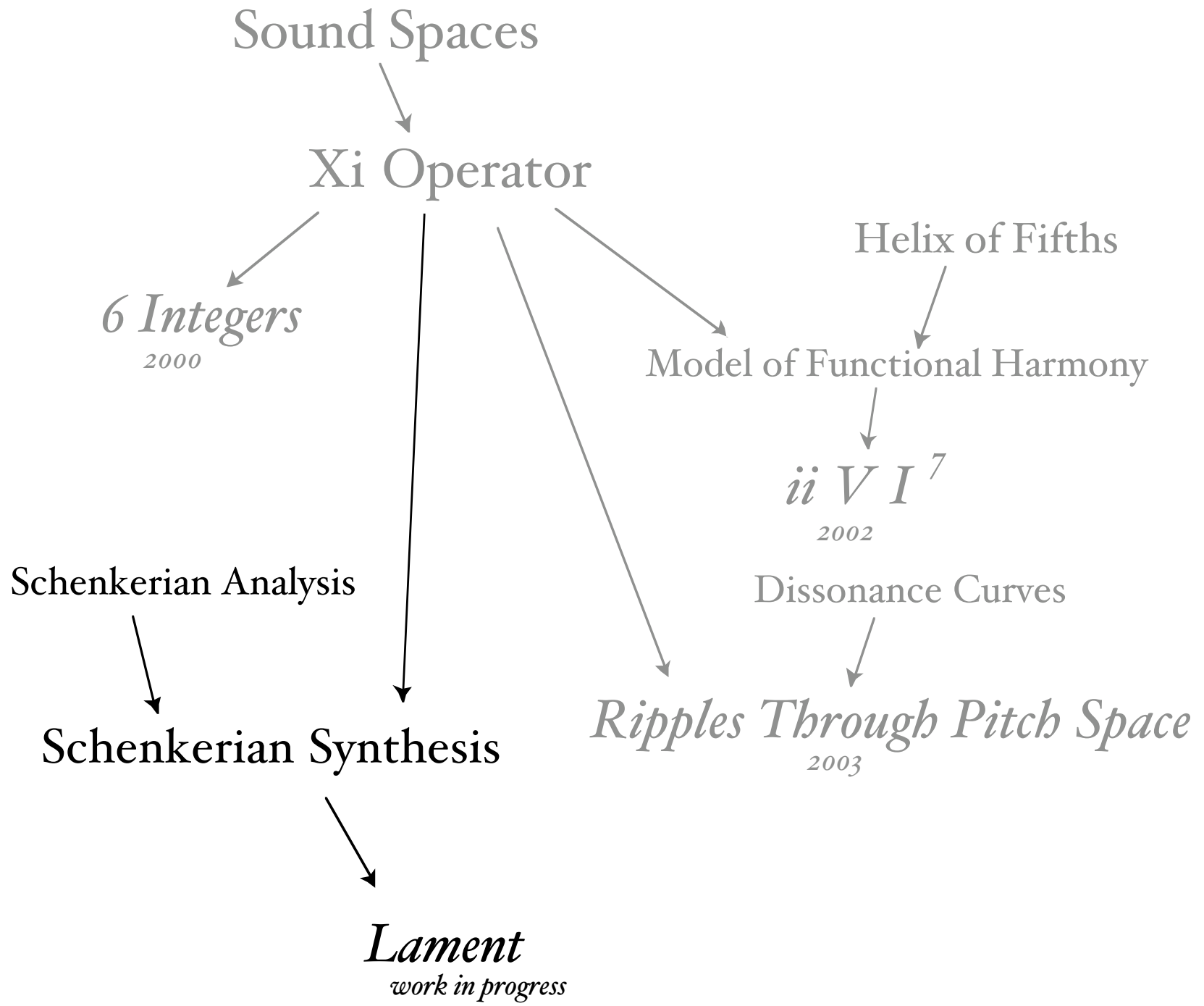
# *Ripples Through Pitch Space*

2003

4 Mvts

20 evenly spaced particles





*But what about melodies?*

*But what about melodies?*

Idea: Do Schenkerian analysis in  
reverse via Xi *Schenkerian synthesis!*

*But what about melodies?*

Idea: Do Schenkerian analysis in  
reverse via Xi *Schenkerian synthesis!*

*But what is  
Schenkerian analysis?*

# Introduction to Schenkerian Analysis in One Page!

# Introduction to Schenkerian Analysis in One Page!

## *Happy Birthday!*









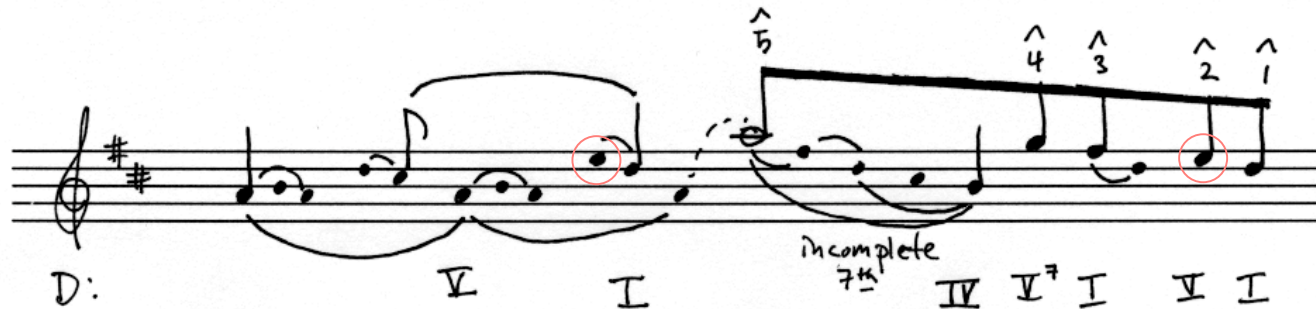


# Introduction to Schenkerian Analysis in One Page!

## *Happy Birthday!*



*Relative structural significance?*

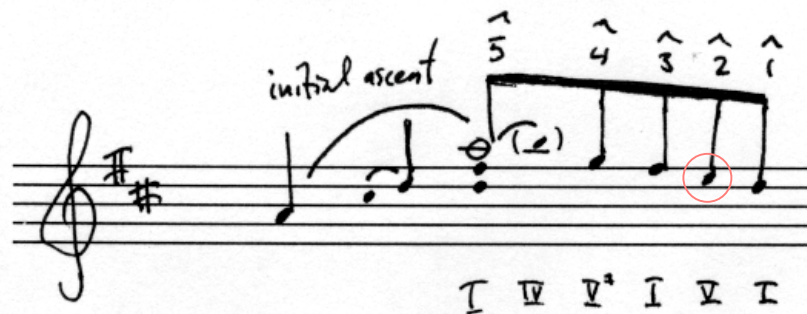
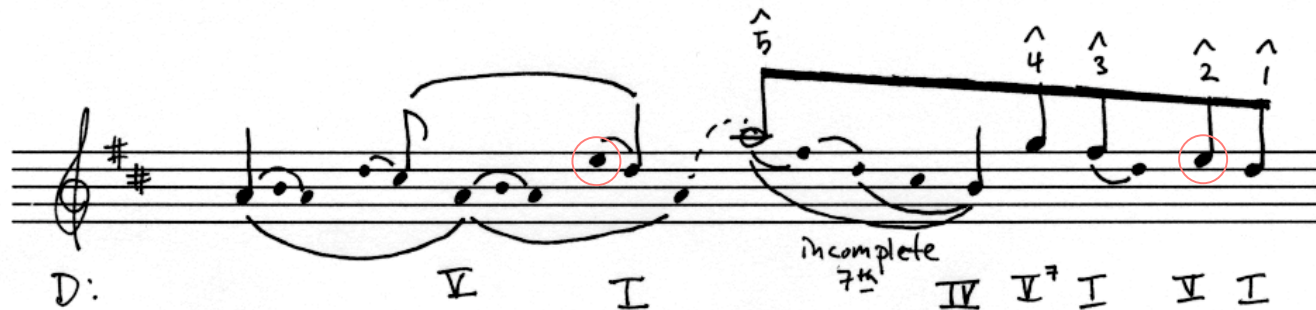


# Introduction to Schenkerian Analysis in One Page!

## *Happy Birthday!*



*Relative structural significance?*



# Inheritance Functions for Schenkerian Synthesis

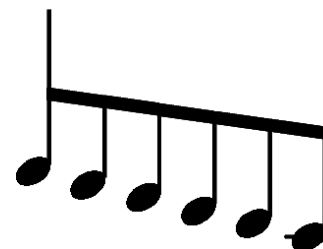
*ascending*

*descending*

*to*

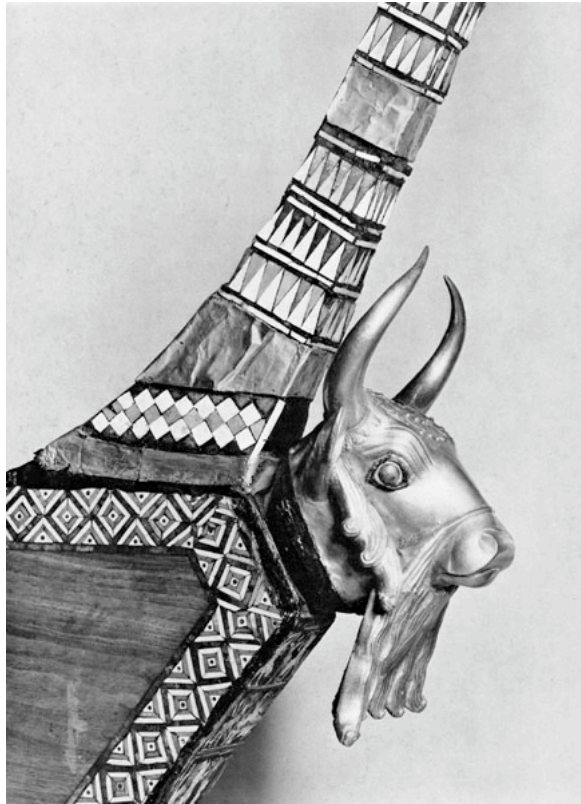


*from*



# *Lament*

*work in progress*



“Lyre from Ur”  
from ca. 2400 B.C.  
Source: Oriental Institute

2 Mvts  
*so far*

Melodic line created with  
Schenkerian Synthesis: embedded  
within self.

Sound Spaces

Xi Operator

*6 Integers*  
2000

Helix of Fifths  
Model of Functional Harmony

*ii V I*<sup>7</sup>  
2002

Schenkerian Analysis

Dissonance Curves

Schenkerian Synthesis

*Ripples Through Pitch Space*  
2003

*Lament*  
work in progress

## *Summary:*

Mathematical structures were described which can be used to produce music through computation. Most important was the versatile Xi Operator, which may be used to construct models for expressive rhythm, functional harmony and melody.



*Please visit my web page*

[www.mctague.org/carl](http://www.mctague.org/carl)

*to hear these pieces and others.*





Want the mathematical structures to be *musically meaningful* whatever that means at least inspired or informed by musical experience, intuition or theory.

Can also use  $[0,1] \rightarrow \mathbb{R}$  to control  
continuous parameters of sound.

*E.g. loudness*



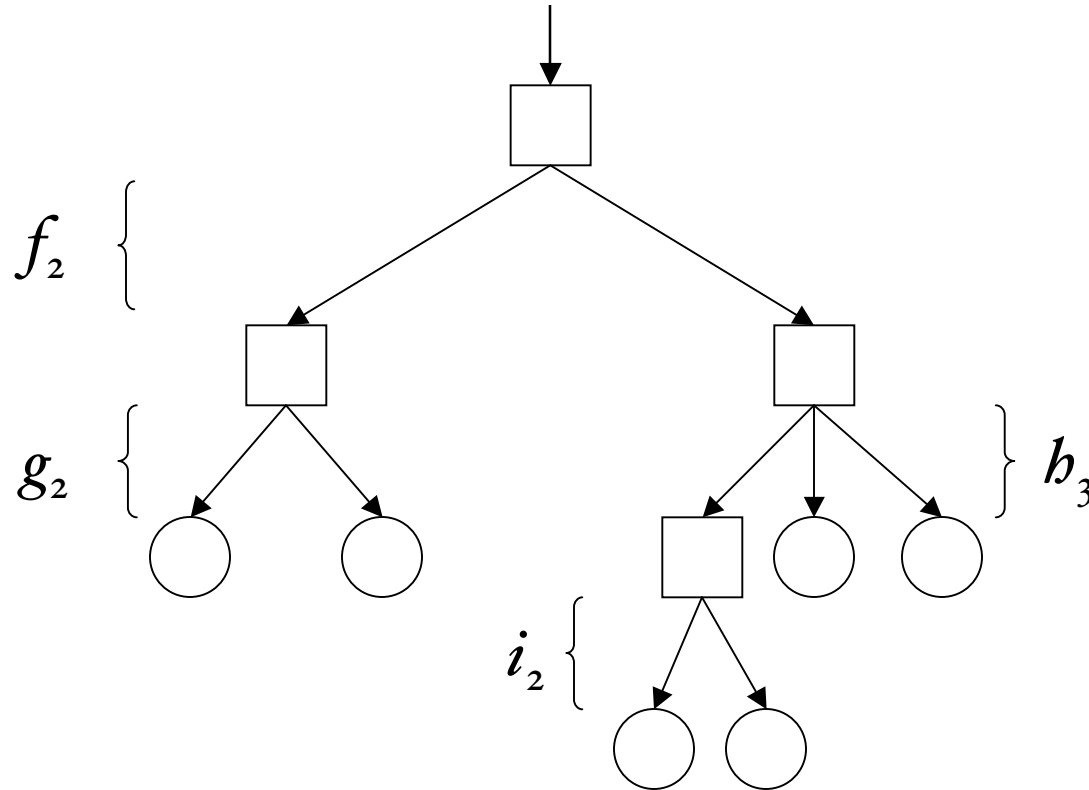
I call this construction the  
*Xi Operator*  $\Xi$

Given a family of inheritance functions  
and an ordered list of sound spaces, it  
produces a new sound space:

$$\left( \begin{array}{l} \{f_n : A \square B^n\}_n \\ (c_i : B \square (L^p)^m)_{i=1, \dots, N} \end{array} \right) \xrightarrow{\Xi} A \square (L^p)^m$$

# An alternate view;

*inductive use of  $X_i$  as information propagating through a tree:*



Information flows down the tree, manipulated at each branch by the local inheritance function until it reaches the Os, which denote possibly distinct, existing sound spaces.